

# THE AUTOMORPHISM GROUP OF A SUPERSINGULAR $K3$ SURFACE WITH ARTIN INVARIANT 1 IN CHARACTERISTIC 3

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ABSTRACT. We present a finite set of generators of the automorphism group of a supersingular  $K3$  surface with Artin invariant 1 in characteristic 3.

## 1. INTRODUCTION

To determine the automorphism group  $\text{Aut}(Y)$  of a given algebraic  $K3$  surface  $Y$  is an important problem. It follows from the Torelli type theorem for  $K3$  surfaces defined over  $\mathbb{C}$  (Piatetskii-Shapiro, Shafarevich [27]) that  $\text{Aut}(Y)$  is isomorphic to  $O(S_Y)/W(S_Y)^{(-2)}$  up to finite groups where  $S_Y$  is the Picard lattice of  $Y$ ,  $O(S_Y)$  is the orthogonal group of  $S_Y$  and  $W(S_Y)^{(-2)}$  is the normal subgroup of  $O(S_Y)$  generated by  $(-2)$ -reflections. In particular  $\text{Aut}(Y)$  is finite if and only if  $S_Y$  is  $(-2)$ -reflective, that is,  $W(S_Y)^{(-2)}$  is of finite index in  $O(S_Y)$ . All  $(-2)$ -reflective lattices are classified (Nikulin [22], [23], Vinberg [50]). On the other hand, Shioda-Inose [43] showed that  $\text{Aut}(Y)$  is infinite for all *singular*  $K3$  surfaces, that is, complex  $K3$  surfaces with the maximum Picard number 20. In case  $\text{Aut}(Y)$  is infinite, it is, in general, very difficult to describe it explicitly. If  $S_Y$  is *reflective*, that is, the *full* reflection group  $W(S_Y)$  generated by not only  $(-2)$ -reflections but also all reflections in  $O(S_Y)$  is of finite index in  $O(S_Y)$ , then one may find a fundamental domain of  $W(S_Y)$  (The reflective lattices in rank 3 are classified in Allcock [1]). By this method, Vinberg [49] gave a concrete description of  $\text{Aut}(Y)$  for two singular  $K3$  surfaces. However, in case that  $S_Y$  is not reflective, there were no general methods to study  $\text{Aut}(Y)$ . In case that  $S_Y$  is non-reflective, the first author [17] gave a set of generators of the automorphism group of a generic Jacobian Kummer surface by applying the results of Conway [8] and Borchers [5, 6] on the reflection group of the even unimodular lattice of signature  $(1, 25)$ .

In this paper, by using the same method, we present a set of generators of the automorphism group of a supersingular  $K3$  surface  $X$  in characteristic 3 with Artin invariant 1. We note that  $S_X$  is not reflective. Our method is computational, and relies heavily on computer-aided calculation. It gives us generators in explicit form, and it can be easily applied to many other  $K3$  surfaces by modifying computer programs.

A  $K3$  surface defined over an algebraically closed field  $k$  is said to be *supersingular* (in the sense of Shioda) if its Picard number is 22. Supersingular  $K3$  surfaces exist only when  $k$  is of positive characteristic. Let  $Y$  be a supersingular  $K3$  surface in characteristic  $p > 0$ , and let  $S_Y$  denote its Néron-Severi lattice. Artin [4] showed that the discriminant group of  $S_Y$  is a  $p$ -elementary abelian group of rank  $2\sigma$ , where  $\sigma$  is an integer such that  $1 \leq \sigma \leq 10$ . This integer  $\sigma$  is called the *Artin invariant* of  $Y$ . Ogus [24, 25] proved that a supersingular  $K3$  surface with Artin invariant 1 in characteristic  $p$  is unique up to isomorphisms and the Torelli type theorem for them in characteristic  $p > 2$  (see also Rudakov, Shafarevich [28]).

In the following, we consider the Fermat quartic surface

$$X := \{w^4 + x^4 + y^4 + z^4 = 0\} \subset \mathbb{P}^3$$

defined over an algebraically closed field  $k$  of characteristic 3, which is a supersingular  $K3$  surface with Artin invariant 1. Let

$$h_0 := [\mathcal{O}_X(1)] \in S_X$$

denote the class of the hyperplane section of  $X$ . The projective automorphism group  $\text{Aut}(X, h_0)$  of  $X \subset \mathbb{P}^3$  is equal to the finite subgroup  $\text{PGU}_4(\mathbb{F}_9)$  of  $\text{PGL}_4(k)$  with order 13, 063, 680. Segre [33] showed that  $X$  contains 112 lines. (In characteristic 0, the Fermat quartic surface contains only 48 lines and its projective automorphism group is of order 1536. See Segre [32].) Tate [47] showed that  $X$  is supersingular by means of representations of  $\text{PGU}_4(\mathbb{F}_9)$ . Mizukami [19] showed that the classes of these 112 lines form a lattice of discriminant  $-9$  (see Schütt, Shioda, van Luijk [31]). Hence these classes span  $S_X$  and the Artin invariant of  $X$  is 1 (see also Shioda [44] and [45]).

Let  $(w, x, y)$  be the affine coordinates of  $\mathbb{P}^3$  with  $z = 1$ , and let  $F_{1j}$  and  $F_{2j}$  be polynomials of  $(w, x, y)$  with coefficients in

$$\mathbb{F}_9 = \mathbb{F}_3(i) = \{0, \pm 1, \pm i, \pm(1+i), \pm(1-i)\}, \quad \text{where } i := \sqrt{-1},$$

given in Table 1.1.

**Proposition 1.1.** *For  $\nu = 1$  and 2, the rational map*

$$(w, x, y) \mapsto [F_{\nu 0} : F_{\nu 1} : F_{\nu 2}] \in \mathbb{P}^2$$

*induces a morphism  $\phi_\nu : X \rightarrow \mathbb{P}^2$  of degree 2.*

We denote by

$$X \xrightarrow{\psi_\nu} Y_\nu \xrightarrow{\pi_\nu} \mathbb{P}^2$$

the Stein factorization of  $\phi_\nu : X \rightarrow \mathbb{P}^2$ , and let  $B_\nu \subset \mathbb{P}^2$  be the branch curve of the finite morphism  $\pi_\nu : Y_\nu \rightarrow \mathbb{P}^2$  of degree 2. Note that  $Y_\nu$  is a normal  $K3$  surface, and hence  $Y_\nu$  has only rational double points as its singularities (see Artin [2, 3]). Let  $[x_0 : x_1 : x_2]$  be the homogeneous coordinates of  $\mathbb{P}^2$ .

$$\begin{aligned}
F_{10} &= (1+i) + (1+i)w + (1-i)x - y - (1-i)wx - x^2 + iwy \\
&\quad + ixy - iy^2 + (1+i)w^3 - iw^2x + (1+i)wx^2 - ix^3 + w^2y \\
&\quad + (1+i)wxy + (1+i)x^2y - (1-i)wy^2 - (1+i)xy^2 + iy^3 \\
F_{11} &= (1-i) - (1+i)x - (1-i)y - (1-i)w^2 - (1-i)wx - (1-i)x^2 \\
&\quad - (1+i)wy - xy - (1+i)y^2 - w^3 + (1-i)w^2x + wx^2 - ix^3 \\
&\quad - (1+i)w^2y - (1+i)wxy + x^2y - iwy^2 - xy^2 + (1-i)y^3 \\
F_{12} &= (1+i)w - ix - y - w^2 - wx - ix^2 - ixy + iy^2 + iw^3 \\
&\quad - (1+i)wx^2 + ix^3 - iw^2y - wxy + (1-i)wy^2 + (1+i)y^3 \\
\hline
F_{20} &= -1 - iw + (1+i)x - y - (1+i)w^2 - wx - (1-i)x^2 - iwy + (1+i)xy \\
&\quad - (1-i)w^3 + w^2x - wx^2 + x^3 - w^2y + (1-i)wxy + x^2y + (1-i)wy^2 \\
&\quad + (1-i)xy^2 + (1+i)y^3 - w^3x - iw^2x^2 - wx^3 + w^3y - (1+i)w^2xy \\
&\quad - (1-i)wxy^2 + x^2y^2 - (1-i)wy^3 - (1+i)xy^3 - y^4 + (1-i)w^3x^2 - ix^5 \\
&\quad + (1-i)w^3xy + (1+i)wx^3y - iw^3y^2 + (1+i)w^2xy^2 - (1+i)wx^2y^2 \\
&\quad + ix^3y^2 - w^2y^3 - (1+i)wxy^3 - (1-i)x^2y^3 + iwy^4 + (1-i)xy^4 + (1+i)y^5 \\
F_{21} &= -(1-i) + iw + (1-i)y - (1+i)w^2 + wx + (1+i)x^2 + (1+i)wy - (1+i)xy \\
&\quad - iy^2 - w^3 + iw^2x + (1+i)wx^2 - x^3 - (1+i)w^2y - (1-i)wxy - (1-i)x^2y \\
&\quad - iwy^2 - (1+i)xy^2 + y^3 - (1-i)w^3x - wx^3 + (1-i)x^4 + (1-i)w^3y + iw^2xy \\
&\quad + (1-i)wx^2y - ix^3y + (1-i)w^2y^2 + (1-i)wxy^2 - (1+i)x^2y^2 + (1-i)wy^3 \\
&\quad - ixy^3 + iy^4 + w^3x^2 + w^2x^3 + (1-i)wx^4 - ix^5 - iw^3xy + w^2x^2y + (1+i)wx^3y \\
&\quad + x^4y + w^3y^2 - w^2xy^2 - wx^2y^2 + iw^2y^3 + (1+i)wxy^3 - iwy^4 - ixy^4 + y^5 \\
F_{22} &= (1-i) - (1+i)w - (1+i)x - (1-i)y + iw^2 - (1+i)wx - (1-i)x^2 + iwy \\
&\quad - (1+i)xy - w^3 - iw^2x - wx^2 + x^3 - (1-i)w^2y + wxy + x^2y + (1+i)wy^2 \\
&\quad - (1+i)xy^2 - y^3 + iw^3x - (1-i)w^2x^2 - wx^3 - (1+i)x^4 + iw^3y + w^2xy \\
&\quad + (1-i)wx^2y - (1-i)w^2y^2 + (1+i)wxy^2 + iwy^3 + xy^3 + (1-i)y^4 - iw^3x^2 \\
&\quad - (1+i)wx^4 + x^5 - (1-i)w^3xy - iw^2x^2y + (1+i)wx^3y + (1-i)x^4y - w^3y^2 \\
&\quad - (1+i)w^2xy^2 + iw^2x^2y^2 + ix^3y^2 - wxy^3 - (1-i)x^2y^3 - wy^4 - xy^4 - y^5
\end{aligned}$$

TABLE 1.1. Polynomials  $F_{1j}$  and  $F_{2j}$ 

**Proposition 1.2.** (1) *The ADE-type of the singularities of  $Y_1$  is  $6A_1 + 4A_2$ . The branch curve  $B_1$  is defined by  $f_1 = 0$ , where*

$$\begin{aligned}
f_1 &:= x_0^6 + x_0^5x_1 - x_0^3x_1^3 - x_0x_1^5 - x_0^4x_2^2 \\
&\quad + x_0x_1^3x_2^2 + x_1^4x_2^2 + x_0^2x_2^4 + x_1^2x_2^4 + x_2^6.
\end{aligned}$$

(2) *The ADE-type of the singularities of  $Y_2$  is  $A_1 + A_2 + 2A_3 + 2A_4$ . The branch curve  $B_2$  is defined by  $f_2 = 0$ , where*

$$\begin{aligned}
f_2 &:= x_0^5x_1 + x_0^2x_1^4 - x_0^4x_2^2 + x_0x_1^3x_2^2 \\
&\quad + x_1^4x_2^2 - x_0^2x_2^4 - x_0x_1x_2^4 - x_1^2x_2^4 - x_2^6.
\end{aligned}$$

Our main result is as follows:

**Theorem 1.3.** *Let  $g_\nu \in \text{Aut}(X)$  denote the involution induced by the deck transformation of  $\pi_\nu : Y_\nu \rightarrow \mathbb{P}^2$ . Then  $\text{Aut}(X)$  is generated by  $\text{Aut}(X, h_0) = \text{PGU}_4(\mathbb{F}_9)$  and  $g_1, g_2$ .*

See Theorem 7.1 for a more explicit description of the involutions  $g_1$  and  $g_2$ .

Let  $L$  denote an even unimodular lattice of rank 26 with signature  $(1, 25)$ , which is unique up to isomorphisms by Eichler's theorem. Conway [8] determined the fundamental domain in a positive cone of  $L \otimes \mathbb{R}$  under the action of  $W(L)^{(-2)}$ . Borcherds [5, 6] applied Conway theory to the investigation of the orthogonal groups of even hyperbolic lattices  $S$  primitively embedded in  $L$  with the orthogonal complement  $T$  being a (negative definite) root lattice. He obtained a chamber decomposition of a positive cone of  $S \otimes \mathbb{R}$  by restricting Conway's chamber decomposition of a positive cone of  $L \otimes \mathbb{R}$ .

We employ Borcherds' method in our case. We take the root lattice  $A_2 \oplus A_2$  as  $T$ . Then the orthogonal complement of  $T$  in  $L$  is isomorphic to the Néron-Severi lattice  $S_X$ . Let  $\mathcal{P}_{S_X}$  denote the connected component of  $\{x \in S_X \otimes \mathbb{R} \mid x^2 > 0\}$  that contains  $h_0$ . We prove Theorem 1.3 by calculating a closed chamber  $D_{S_0}$  in the cone  $\mathcal{P}_{S_X}$  with the following properties (see Section 6):

- (1) The chamber  $D_{S_0}$  is invariant under the action of  $\text{Aut}(X, h_0)$ .
- (2) For any nef class  $v \in S_X$ , there exists  $\gamma \in \text{Aut}(X)$  such that  $v^\gamma \in D_{S_0}$ .
- (3) For nef classes  $v, v'$  in the interior of  $D_{S_0}$ , there exists  $\gamma \in \text{Aut}(X)$  such that  $v' = v^\gamma$  if and only if there exists  $\tau \in \text{Aut}(X, h_0)$  such that  $v' = v^\tau$ .

This chamber  $D_{S_0}$  is bounded by  $112 + 648 + 5184$  hyperplanes in  $\mathcal{P}_{S_X}$ . See Proposition 4.5 for the explicit description of these walls. Using  $D_{S_0}$  and these walls, we can also present a finite set of generators of  $O^+(S_X)$  (see Theorem 8.2). This chamber  $D_{S_0}$  is not only important for the calculation of  $\text{Aut}(X)$ , but also useful for the classification of geometric objects on  $X$  modulo  $\text{Aut}(X)$ . See Corollary 6.2 for an example.

As mentioned above, the first author [17] determined the automorphism group of a generic Jacobian Kummer surface by embedding its Néron-Severi lattice into  $L$  primitively with the orthogonal complement being a root lattice of type  $6A_1 + A_3$  (The description of the automorphism group was applied by Kumar [18] to obtain all elliptic fibrations with a section up to automorphisms).

Keum and the first author [16] applied the same method to the Kummer surface of the product of two elliptic curves, Dolgachev and Keum [13] applied it to quartic Hessian surfaces, and Dolgachev and the first author [12] applied it to the super-singular  $K3$  surface in characteristic 2 with Artin invariant 1. We remark that the Picard lattice of the last  $K3$  surface is reflective.

There are many  $K3$  surfaces  $Y$  such that  $S_Y$  is isomorphic to the orthogonal complement to a primitive root sublattice of  $L$ ; for example, a complex  $K3$  surface

whose Picard lattice is isomorphic to  $U \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$ , which is obtained as the orthogonal complement of the root lattice  $D_7$  in  $L$ . We hope that, for these  $K3$  surfaces, we can obtain a set of generators of  $\text{Aut}(Y)$  by modifying our computer programs.

If the orthogonal complement of  $S_Y$  embedded in  $L$  is not a root lattice, then this method does not always work. Consider, for example, the Fermat quartic surface in characteristic 0 or a supersingular  $K3$  surface with Artin invariant 1 in characteristic 5. We can embed  $S_Y$  into  $L$  primitively, but the orthogonal complement is *not* a root lattice. An experimental computation shows that many isomorphism classes of chambers in  $S_Y \otimes \mathbb{R}$  are obtained as the intersections with the Conway chambers in  $L \otimes \mathbb{R}$ . However, for example, in case of Fermat quartic surface in characteristic 0, we obtain 48 faces of a chamber corresponding to 48 lines on the Fermat quartic. On the other hand, a supersingular  $K3$  surface with Artin invariant 1 in characteristic 5 is obtained as the double cover of  $\mathbb{P}^2$  branched along the Fermat sextic curve. The pullback of 126 lines on  $\mathbb{P}^2(\mathbb{F}_{25})$  are 252 smooth rational curves on the  $K3$  surface. In this case, we have these 252 smooth rational curves as 252 faces of a chamber. Moreover the group  $\text{PGU}(3, \mathbb{F}_5)$  appears as the group of automorphisms of the chamber. By a similar way, we can see that the sporadic simple groups  $M_{11}$  and  $M_{22}$  called the Mathieu groups act on a supersingular  $K3$  surface in characteristic 11 with Artin invariant 1. Note that the last  $K3$  surface is isomorphic to the Fermat quartic surface in characteristic 11 (see Shioda [44] and [45]).

The new idea introduced in this paper is that, in order to find automorphisms of  $X$  necessary to generate  $\text{Aut}(X)$ , we search for polarizations of degree 2 whose classes are located on the walls of the chamber decomposition of the cone  $\mathcal{P}_{S_X}$ . Since  $S_X$  is generated by the classes of lines, and the defining ideals of these lines are easily calculated, we can write the automorphisms of  $X$  as a list of rational functions on  $X$ . The computational tools used in this paper have been developed by the second author for the study [39] of various double plane models of a supersingular  $K3$  surface in characteristic 5 with Artin invariant 1. The computational data for this paper is available from the second author's webpage [40].

In [36] and [38], the second author showed that every supersingular  $K3$  surface in any characteristic with arbitrary Artin invariant is birational to a double cover of the projective plane. In [37], [41] and [26, 39], projective models of supersingular  $K3$  surfaces in characteristic 2, 3 and 5 were investigated, respectively.

The study of the automorphism group of the Fermat quartic surface goes back to Segre [32], in which it was proved that the automorphism group is infinite in characteristic 0. Then Shioda [42] showed that  $\text{Aut}(X)$  contains  $\mathbb{Z}^2$  by showing the existence of an elliptic fibration of rank 2 on  $X$ . Recently, elliptic fibrations on  $X$  was classified in Sengupta [34] by embedding  $S_X$  into  $L$ . On the other hand,

configurations of smooth rational curves on  $X$  was studied in Katsura and the second author [15] with respect to an embedding of  $S_X$  into  $L$ .

This paper is organized as follows. In Section 2, we give a review of the theory of Conway and Borchers, and investigate chamber decomposition induced on a positive cone of a primitive hyperbolic sublattice  $S$  of  $L$ . In Section 3, we give explicitly a basis of the Néron-Severi lattice  $S_X$  of  $X$ , and describe a method to compute the action of  $\text{Aut}(X, h_0)$  on  $S_X$ . In Section 4, we embed  $S_X$  into  $L$ , and study the obtained chamber decomposition in detail. In particular, we investigate the walls of the chamber  $D_{S_0}$  that contains the class  $h_0$ . In Section 5, we prove Propositions 1.1 and 1.2, and show that the involutions  $g_1$  and  $g_2$  map  $h_0$  to its mirror images into walls of the chamber  $D_{S_0}$ . Then we can prove Theorem 1.3 in Section 6. In Section 7, we give another description of the involutions  $g_\nu$ . In the last section, we give a set of generators of  $O^+(S_X)$ .

## 2. LEECH ROOTS

**2.1. Terminologies and notation.** We fix some terminologies and notation about lattices. A *lattice*  $M$  is a free  $\mathbb{Z}$ -module of finite rank with a non-degenerate symmetric bilinear form

$$(\ , \ )_M : M \times M \rightarrow \mathbb{Z}.$$

A submodule  $N$  of  $M$  is said to be *primitive* if  $M/N$  is torsion free. For a submodule  $N$  of  $M$ , we denote by  $N^\perp \subset M$  the submodule defined by

$$N^\perp := \{ u \in M \mid (u, v)_M = 0 \text{ for all } v \in N \},$$

which is primitive by definition. We denote by  $O(M)$  the orthogonal group of  $M$ . Throughout this paper, we let  $O(M)$  act on  $M$  from *right*. Suppose that  $M$  is of rank  $r$ . We say that  $M$  is *hyperbolic* (resp. *negative-definite*) if the signature of the symmetric bilinear form  $(\ , \ )_M$  on  $M \otimes \mathbb{R}$  is  $(1, r-1)$  (resp.  $(0, r)$ ). We define the *dual lattice*  $M^\vee$  of  $M$  by

$$M^\vee := \{ u \in M \otimes \mathbb{Q} \mid (u, v)_M \in \mathbb{Z} \text{ for all } v \in M \}.$$

Then  $M$  is contained in  $M^\vee$  as a submodule of finite index. The finite abelian group  $M^\vee/M$  is called the *discriminant group* of  $M$ . We say that  $M$  is *unimodular* if  $M = M^\vee$ .

A lattice  $M$  is said to be *even* if  $(v, v)_M \in 2\mathbb{Z}$  holds for any  $v \in M$ . The discriminant group  $M^\vee/M$  of an even lattice  $M$  is naturally equipped with the quadratic form

$$q_M : M^\vee/M \rightarrow \mathbb{Q}/2\mathbb{Z}$$

defined by  $q_M(u \bmod M) := (u, u)_M \bmod 2\mathbb{Z}$ . We call  $q_M$  the *discriminant form* of  $M$ . The automorphism group of  $q_M$  is denoted by  $O(q_M)$ . There exists a natural homomorphism  $O(M) \rightarrow O(q_M)$ .

Suppose that  $M$  is hyperbolic. Then the open subset

$$\{ x \in M \otimes \mathbb{R} \mid (x, x)_M > 0 \}$$

of  $M \otimes \mathbb{R}$  has two connected components. A *positive cone* of  $M$  is one of them. We fix a positive cone  $\mathcal{P}$ . The *autochronous orthogonal group*  $O^+(M)$  of  $M$  is the group of isometries of  $M$  that preserve  $\mathcal{P}$ . Then  $O^+(M)$  is a subgroup of  $O(M)$  with index 2. Note that  $O^+(M)$  acts on  $\mathcal{P}$ . For a nonzero vector  $u \in M \otimes \mathbb{R}$ , we denote by  $(u)_M^\perp$  the hyperplane of  $M \otimes \mathbb{R}$  defined by

$$(u)_M^\perp := \{ x \in M \otimes \mathbb{R} \mid (x, u)_M = 0 \}.$$

Let  $\mathcal{R}$  be a set of non-zero vectors of  $M \otimes \mathbb{R}$ , and let

$$\mathcal{H} := \{ (u)_M^\perp \mid u \in \mathcal{R} \}$$

be the family of hyperplanes defined by  $\mathcal{R}$ . Suppose that  $\mathcal{H}$  is locally finite in  $\mathcal{P}$ . Then the closure in  $\mathcal{P}$  of each connected component of

$$\mathcal{P} \setminus \left( \mathcal{P} \cap \bigcup_{u \in \mathcal{R}} (u)_M^\perp \right)$$

is called an  $\mathcal{R}$ -chamber. Let  $D$  be an  $\mathcal{R}$ -chamber. We denote by  $D^\circ$  the interior of  $D$ . We say that a hyperplane  $(u)_M^\perp \in \mathcal{H}$  *bounds*  $D$ , or that  $(u)_M^\perp$  is a *wall* of  $D$ , if  $(u)_M^\perp \cap D$  contains a non-empty open subset of  $(u)_M^\perp$ . We denote the set of walls of  $D$  by

$$\mathcal{W}(D) := \{ (u)_M^\perp \in \mathcal{H} \mid (u)_M^\perp \text{ bounds } D \}.$$

Suppose that  $\mathcal{R}$  is invariant under  $u \mapsto -u$ . We choose a point  $p \in D^\circ$ , and put

$$\widetilde{\mathcal{W}}(D) := \{ u \in \mathcal{R} \mid (u)_M^\perp \text{ bounds } D \text{ and } (u, p)_M > 0 \},$$

which is independent of the choice of  $p$ . It is obvious that  $D$  is equal to

$$\{ x \in \mathcal{P} \mid (x, u)_M \geq 0 \text{ for all } u \in \widetilde{\mathcal{W}}(D) \}.$$

**2.2. Conway theory.** We review the theory of Conway [8]. Let  $L$  be an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphisms by Eichler's theorem (see, for example, [7, Chapter 11, Theorem 1.4]). We choose and fix a positive cone  $\mathcal{P}_L$  once and for all. A vector  $r \in L$  is called a *root* if the reflection  $s_r : L \otimes \mathbb{R} \rightarrow L \otimes \mathbb{R}$  defined by

$$x \mapsto x - \frac{2(x, r)_L}{(r, r)_L} \cdot r$$

preserves  $L$  and  $\mathcal{P}_L$ , or equivalently, if  $(r, r)_L = -2$ . We denote by  $\mathcal{R}_L$  the set of roots of  $L$ , which is invariant under  $r \mapsto -r$ . Let  $W(L)$  denote the subgroup of  $O^+(L)$  generated by the reflections  $s_r$  associated with all the roots  $r \in \mathcal{R}_L$ . Then  $W(L)$  is a normal subgroup of  $O^+(L)$ . The family of hyperplanes

$$\mathcal{H}_L := \{ (r)_L^\perp \mid r \in \mathcal{R}_L \}$$

is locally finite in  $\mathcal{P}_L$ . Hence we can consider  $\mathcal{R}_L$ -chambers. By definition, each  $\mathcal{R}_L$ -chamber is a fundamental domain of the action of  $W(L)$  on  $\mathcal{P}_L$ .

A non-zero primitive vector  $w \in L$  is called a *Weyl vector* if  $(w, w)_L = 0$ ,  $w$  is contained in the closure of  $\mathcal{P}_L$  in  $L \otimes \mathbb{R}$ , and the negative-definite even unimodular lattice  $\langle w \rangle^\perp / \langle w \rangle$  of rank 24 has no vectors of square norm  $-2$ . Let  $w \in L$  be a Weyl vector. We put

$$LR(w) := \{ r \in \mathcal{R}_L \mid (w, r)_L = 1 \}.$$

A root in  $LR(w)$  is called a *Leech root with respect to  $w$* .

Suppose that  $w$  is a non-zero primitive vector of norm 0 contained in the closure of  $\mathcal{P}_L$ . Then there exists a vector  $w' \in L$  such that  $(w, w')_L = 1$  and  $(w', w')_L = 0$ . Let  $U \subset L$  denote the hyperbolic sublattice of rank 2 generated by  $w$  and  $w'$ . By Niemeier's classification [20] of even definite unimodular lattices of rank 24 (see also [11, Chapter 18]), we see that the condition that  $\langle w \rangle^\perp / \langle w \rangle$  have no vectors of square norm  $-2$  is equivalent to the condition that the orthogonal complement  $U^\perp$  of  $U$  in  $L$  be isomorphic to the (negative-definite) Leech lattice  $\Lambda$ . From this fact, we can deduce the following:

**Proposition 2.1.** *The group  $O^+(L)$  acts on the set of Weyl vectors transitively.*

**Proposition 2.2.** *Suppose that  $w$  is a Weyl vector and that  $w' \in L$  satisfies  $(w, w')_L = 1$  and  $(w', w')_L = 0$ . Via an isomorphism  $\rho : \Lambda \xrightarrow{\sim} U^\perp$ , the map*

$$\lambda \mapsto -\frac{2 + (\lambda, \lambda)_\Lambda}{2}w + w' + \rho(\lambda)$$

*induces a bijection from the Leech lattice  $\Lambda$  to the set  $LR(w)$ .*

Using Vinberg's algorithm [48] and the result on the covering radius of the Leech lattice [10], Conway [8] proved the following:

**Theorem 2.3.** *Let  $w \in L$  be a Weyl vector. Then*

$$D_L(w) := \{ x \in \mathcal{P}_L \mid (x, r)_L \geq 0 \text{ for all } r \in LR(w) \}$$

*is an  $\mathcal{R}_L$ -chamber, and  $\widetilde{W}(D_L(w))$  is equal to  $LR(w)$ ; that is,  $(r)_L^\perp$  bounds  $D_L(w)$  for any  $r \in LR(w)$ . The map  $w \mapsto D_L(w)$  is a bijection from the set of Weyl vectors to the set of  $\mathcal{R}_L$ -chambers.*

*Remark 2.4.* Using Proposition 2.2, Conway [8] also showed that the automorphism group  $\text{Aut}(D_L(w)) \subset O^+(L)$  of an  $\mathcal{R}_L$ -chamber  $D_L(w)$  is isomorphic to the group  $\cdot\infty$  of affine automorphisms of the Leech lattice  $\Lambda$ . Hence  $O^+(L)$  is isomorphic to the split extension of  $\cdot\infty$  by  $W(L)$ .



**2.3. Restriction of  $\mathcal{R}_L$ -chambers to a primitive sublattice.** Let  $S$  be an even hyperbolic lattice of rank  $r < 26$  primitively embedded in  $L$ . Following Borcherds [5, 6], we explain how the Leech roots of  $L$  induce a chamber decomposition on the positive cone

$$\mathcal{P}_S := \mathcal{P}_L \cap (S \otimes \mathbb{R})$$

of  $S \otimes \mathbb{R}$ .

The orthogonal complement  $T := S^\perp$  of  $S$  in  $L$  is negative-definite of rank  $26 - r$ , and we have

$$S \oplus T \subset L \subset S^\vee \oplus T^\vee$$

with  $[L : S \oplus T] = [S^\vee \oplus T^\vee : L]$ . The projections  $L \otimes \mathbb{R} \rightarrow S \otimes \mathbb{R}$  and  $L \otimes \mathbb{R} \rightarrow T \otimes \mathbb{R}$  are denoted by

$$x \mapsto x_S \quad \text{and} \quad x \mapsto x_T,$$

respectively. Note that, if  $v \in L$ , then  $v_S \in S^\vee$  and  $v_T \in T^\vee$ .

Let  $r \in L$  be a root. Then the hyperplane  $(r)_L^\perp$  contains  $S \otimes \mathbb{R}$  if and only if  $r_S = 0$ , or equivalently, if and only if  $r \in T$ . Since  $T$  is negative-definite, the set

$$\mathcal{R}_T := \{ v \in T \mid (v, v)_T = -2 \}$$

is finite, and therefore there exist only finite number of hyperplanes  $(r)_L^\perp$  that contain  $S \otimes \mathbb{R}$ . Suppose that  $r_S \neq 0$ . If  $(r_S, r_S)_S \geq 0$ , then either  $\mathcal{P}_S$  is entirely contained in the interior of the halfspace

$$\{ x \in L \otimes \mathbb{R} \mid (x, r)_L \geq 0 \}$$

or is disjoint from this halfspace. Hence the hyperplane

$$(r_S)_S^\perp = (r)_L^\perp \cap (S \otimes \mathbb{R})$$

of  $S \otimes \mathbb{R}$  intersects  $\mathcal{P}_S$  if and only if  $(r_S, r_S)_S < 0$ . We put

$$\begin{aligned} \mathcal{R}_S &:= \{ r_S \mid r \in \mathcal{R}_L \text{ and } (r_S, r_S)_S < 0 \} \\ &= \{ r_S \mid r \in \mathcal{R}_L \text{ and } (r_S)_S^\perp \cap \mathcal{P}_S \neq \emptyset \}. \end{aligned}$$

Then the associated family of hyperplanes

$$\mathcal{H}_S := \{ (r_S)_S^\perp \mid r_S \in \mathcal{R}_S \}$$

is locally finite in  $\mathcal{P}_S$ , and hence we can consider  $\mathcal{R}_S$ -chambers in  $\mathcal{P}_S$ . Note that  $\mathcal{R}_S$  is invariant under  $r_S \mapsto -r_S$ . We investigate the relation between  $\mathcal{R}_S$ -chambers and  $\mathcal{R}_L$ -chambers.

If  $D_S \subset \mathcal{P}_S$  is an  $\mathcal{R}_S$ -chamber, then there exists an  $\mathcal{R}_L$ -chamber  $D_L(w) \subset \mathcal{P}_L$  such that  $D_S = D_L(w) \cap (S \otimes \mathbb{R})$  holds. For a given  $\mathcal{R}_S$ -chamber  $D_S$ , the set of  $\mathcal{R}_L$ -chambers  $D_L(w)$  satisfying  $D_S = D_L(w) \cap (S \otimes \mathbb{R})$  is in one-to-one correspondence with the set of connected components of

$$(T \otimes \mathbb{R}) \setminus \bigcup_{r \in \mathcal{R}_T} (r)_T^\perp.$$

Conversely, suppose that an  $\mathcal{R}_L$ -chamber  $D_L(w)$  is given.

**Definition 2.5.** We say that  $D_L(w)$  is  $S$ -nondegenerate if  $D_L(w) \cap (S \otimes \mathbb{R})$  is an  $\mathcal{R}_S$ -chamber.

By definition,  $D_L(w)$  is  $S$ -nondegenerate if and only if  $w$  satisfies the following two conditions:

- (i) There exists  $v \in \mathcal{P}_S$  such that  $(v, r)_L \geq 0$  holds for any  $r \in LR(w)$ .
- (ii) There exists  $v' \in \mathcal{P}_S$  such that  $(v', r)_L > 0$  holds for any  $r \in LR(w)$  with  $(r_S, r_S)_S < 0$ .

If  $D_S = D_L(w) \cap (S \otimes \mathbb{R})$  is an  $\mathcal{R}_S$ -chamber, then  $\widetilde{\mathcal{W}}(D_S)$  is contained in the image of the set

$$LR(w, S) := \{ r \in LR(w) \mid r_S \in \mathcal{R}_S \} = \{ r \in LR(w) \mid (r_S, r_S)_S < 0 \}$$

by the projection  $L \rightarrow S^\vee$ . The following proposition shows that  $D_S$  is bounded by a finite number of walls if  $w_T \neq 0$ , and its proof indicates an effective procedure to calculate  $LR(w, S)$ . (See [39, Section 3] for the details of the necessary algorithms.)

**Proposition 2.6.** *Let  $w \in L$  be a Weyl vector such that  $w_T \neq 0$ . Then  $LR(w, S)$  is a finite set.*

*Proof.* Since  $T$  is negative-definite and  $w_T \neq 0$ , we have

$$(w_S, w_S)_S = -(w_T, w_T)_T > 0.$$

Suppose that  $r \in LR(w)$ . Then we have

$$(w_S, r_S)_S + (w_T, r_T)_T = 1, \quad (r_S, r_S)_S + (r_T, r_T)_T = -2.$$

We have  $(r_S, r_S)_S < 0$  if and only if  $(r_T, r_T)_T > -2$ . Since  $T$  is negative-definite, the set

$$V_T := \{ v \in T^\vee \mid (v, v)_T > -2 \}$$

is finite. For  $v \in V_T$ , we put

$$a_v := 1 - (w_T, v)_T, \quad n_v := -2 - (v, v)_T \quad \text{and} \quad A := \{ (a_v, n_v) \mid v \in V_T \}.$$

For each  $(a, n) \in A$ , we put

$$V_S(a, n) := \{ u \in S^\vee \mid (w_S, u)_S = a, (u, u)_S = n \}.$$

Since  $S$  is hyperbolic and  $(w_S, w_S)_S > 0$ , the set  $V_S(a, n)$  is finite, because  $(\ , \ )_S$  induces on the affine hyperplane

$$\{ x \in S \otimes \mathbb{R} \mid (x, w_S)_S = a \}$$

of  $S \otimes \mathbb{R}$  an inhomogeneous quadratic function whose quadratic part is negative-definite. Then the set  $LR(w, S)$  is equal to

$$L \cap \{ u + v \mid v \in V_T, u \in V_S(a_v, n_v) \},$$

where the intersection is taken in  $S^\vee \oplus T^\vee$ . □

The notion of  $\mathcal{R}_S$ -chamber is useful in the study on  $O^+(S)$  because of the following:

**Proposition 2.7.** *Suppose that the natural homomorphism  $O(T) \rightarrow O(q_T)$  is surjective. Then the action of  $O^+(S)$  preserves  $\mathcal{R}_S$ . In particular, for an  $\mathcal{R}_S$ -chamber  $D_S$  and an isometry  $\gamma \in O^+(S)$ , the image  $D_S^\gamma$  of  $D_S$  by  $\gamma$  is also an  $\mathcal{R}_S$ -chamber. Moreover, if the interior of  $D_S^\gamma$  has a common point with  $D_S$ , then  $D_S^\gamma = D_S$  holds and  $\gamma$  preserves  $\mathcal{W}(D_S)$ .*

*Proof.* By the assumption  $O(T) \twoheadrightarrow O(q_T)$ , every element  $\gamma \in O^+(S)$  lifts to an element  $\tilde{\gamma} \in O(L)$  that satisfies  $\tilde{\gamma}(S) = S$  and  $\tilde{\gamma}|_S = \gamma$  (see [21, Proposition 1.6.1]). Since  $\tilde{\gamma}$  preserves  $\mathcal{R}_L$  and  $\gamma$  preserves  $\mathcal{P}_S$ ,  $\gamma$  preserves  $\mathcal{R}_S$ .  $\square$

### 3. A BASIS OF THE NÉRON-SEVERI LATTICE OF $X$

Recall that  $X \subset \mathbb{P}^3$  is the Fermat quartic surface in characteristic 3. From now on, we put

$$S := S_X,$$

which is an even hyperbolic lattice of rank 22 such that  $S^\vee/S \cong (\mathbb{Z}/3\mathbb{Z})^2$ . We use the affine coordinates  $w, x, y$  of  $\mathbb{P}^3$  with  $z = 1$ .

Note that  $X$  is the Hermitian surface over  $\mathbb{F}_9$  (see [14, Chapter 23]). Hence the number of lines contained in  $X$  is 112 (see [33, n. 32] or [35, Corollary 2.22]). Since the indices of these lines are important throughout this paper, we present defining equations of these lines in Table 3.1. (Note that  $\ell_i \subset X$  implies that  $\ell_i$  is not contained in the plane  $z = 0$  at infinity.) From these 112 lines, we choose the following:

(3.1)

$$\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_9, \ell_{10}, \ell_{11}, \ell_{17}, \ell_{18}, \ell_{19}, \ell_{21}, \ell_{22}, \ell_{23}, \ell_{25}, \ell_{26}, \ell_{27}, \ell_{33}, \ell_{35}, \ell_{49}.$$

The intersection matrix  $N$  of these 22 lines is given in Table 3.2. Since  $\det N = -9$ , the classes  $[\ell_i] \in S$  of the lines  $\ell_i$  in (3.1) form a basis of  $S$ . Throughout this paper, we fix this basis, and write elements of  $S \otimes \mathbb{R}$  as row vectors

$$[x_1, \dots, x_{22}]_S.$$

When we use its dual basis, we write

$$[\xi_1, \dots, \xi_{22}]_S^\vee.$$

Since the hyperplane  $w + (1 + i) = 0$  cuts out from  $X$  the divisor  $\ell_1 + \ell_2 + \ell_3 + \ell_4$ , the class  $h_0 = [\mathcal{O}_X(1)] \in S$  of the hyperplane section is equal to

$$\begin{aligned} h_0 &= [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S \\ &= [1, 1]_S^\vee. \end{aligned}$$

As a positive cone  $\mathcal{P}_S$  of  $S$ , we choose the connected component containing  $h_0$ .

$\ell_1 := \{w + (1+i) = x + (1+i)y = 0\}$	$\ell_2 := \{w + (1+i) = x + (1-i)y = 0\}$
$\ell_3 := \{w + (1+i) = x - (1-i)y = 0\}$	$\ell_4 := \{w + (1+i) = x - (1+i)y = 0\}$
$\ell_5 := \{w + (1-i) = x + (1+i)y = 0\}$	$\ell_6 := \{w + (1-i) = x + (1-i)y = 0\}$
$\ell_7 := \{w + (1-i) = x - (1-i)y = 0\}$	$\ell_8 := \{w + (1-i) = x - (1+i)y = 0\}$
$\ell_9 := \{w - (1-i) = x + (1+i)y = 0\}$	$\ell_{10} := \{w - (1-i) = x + (1-i)y = 0\}$
$\ell_{11} := \{w - (1-i) = x - (1-i)y = 0\}$	$\ell_{12} := \{w - (1-i) = x - (1+i)y = 0\}$
$\ell_{13} := \{w - (1+i) = x + (1+i)y = 0\}$	$\ell_{14} := \{w - (1+i) = x + (1-i)y = 0\}$
$\ell_{15} := \{w - (1+i) = x - (1-i)y = 0\}$	$\ell_{16} := \{w - (1+i) = x - (1+i)y = 0\}$
$\ell_{17} := \{w + iy + i = x + iy - i = 0\}$	$\ell_{18} := \{w + iy + i = x - iy + i = 0\}$
$\ell_{19} := \{w + iy + i = x + y - 1 = 0\}$	$\ell_{20} := \{w + iy + i = x - y + 1 = 0\}$
$\ell_{21} := \{w + iy - i = x + iy + i = 0\}$	$\ell_{22} := \{w + iy - i = x - iy - i = 0\}$
$\ell_{23} := \{w + iy - i = x + y + 1 = 0\}$	$\ell_{24} := \{w + iy - i = x - y - 1 = 0\}$
$\ell_{25} := \{w + iy + 1 = x + iy - 1 = 0\}$	$\ell_{26} := \{w + iy + 1 = x - iy + 1 = 0\}$
$\ell_{27} := \{w + iy + 1 = x + y + i = 0\}$	$\ell_{28} := \{w + iy + 1 = x - y - i = 0\}$
$\ell_{29} := \{w + iy - 1 = x + iy + 1 = 0\}$	$\ell_{30} := \{w + iy - 1 = x - iy - 1 = 0\}$
$\ell_{31} := \{w + iy - 1 = x + y - i = 0\}$	$\ell_{32} := \{w + iy - 1 = x - y + i = 0\}$
$\ell_{33} := \{w - iy + i = x + iy + i = 0\}$	$\ell_{34} := \{w - iy + i = x - iy - i = 0\}$
$\ell_{35} := \{w - iy + i = x + y + 1 = 0\}$	$\ell_{36} := \{w - iy + i = x - y - 1 = 0\}$
$\ell_{37} := \{w - iy - i = x + iy - i = 0\}$	$\ell_{38} := \{w - iy - i = x - iy + i = 0\}$
$\ell_{39} := \{w - iy - i = x + y - 1 = 0\}$	$\ell_{40} := \{w - iy - i = x - y + 1 = 0\}$
$\ell_{41} := \{w - iy + 1 = x + iy + 1 = 0\}$	$\ell_{42} := \{w - iy + 1 = x - iy - 1 = 0\}$
$\ell_{43} := \{w - iy + 1 = x + y - i = 0\}$	$\ell_{44} := \{w - iy + 1 = x - y + i = 0\}$
$\ell_{45} := \{w - iy - 1 = x + iy - 1 = 0\}$	$\ell_{46} := \{w - iy - 1 = x - iy + 1 = 0\}$
$\ell_{47} := \{w - iy - 1 = x + y + i = 0\}$	$\ell_{48} := \{w - iy - 1 = x - y - i = 0\}$
$\ell_{49} := \{w + y + i = x + iy + 1 = 0\}$	$\ell_{50} := \{w + y + i = x - iy - 1 = 0\}$
$\ell_{51} := \{w + y + i = x + y - i = 0\}$	$\ell_{52} := \{w + y + i = x - y + i = 0\}$
$\ell_{53} := \{w + y - i = x + iy - 1 = 0\}$	$\ell_{54} := \{w + y - i = x - iy + 1 = 0\}$
$\ell_{55} := \{w + y - i = x + y + i = 0\}$	$\ell_{56} := \{w + y - i = x - y - i = 0\}$
$\ell_{57} := \{w + y + 1 = x + iy - i = 0\}$	$\ell_{58} := \{w + y + 1 = x - iy + i = 0\}$
$\ell_{59} := \{w + y + 1 = x + y - 1 = 0\}$	$\ell_{60} := \{w + y + 1 = x - y + 1 = 0\}$
$\ell_{61} := \{w + y - 1 = x + iy + i = 0\}$	$\ell_{62} := \{w + y - 1 = x - iy - i = 0\}$
$\ell_{63} := \{w + y - 1 = x + y + 1 = 0\}$	$\ell_{64} := \{w + y - 1 = x - y - 1 = 0\}$
$\ell_{65} := \{w + (1+i)y = x + (1+i) = 0\}$	$\ell_{66} := \{w + (1+i)y = x + (1-i) = 0\}$
$\ell_{67} := \{w + (1+i)y = x - (1-i) = 0\}$	$\ell_{68} := \{w + (1+i)y = x - (1+i) = 0\}$
$\ell_{69} := \{w + (1-i)y = x + (1+i) = 0\}$	$\ell_{70} := \{w + (1-i)y = x + (1-i) = 0\}$
$\ell_{71} := \{w + (1-i)y = x - (1-i) = 0\}$	$\ell_{72} := \{w + (1-i)y = x - (1+i) = 0\}$
$\ell_{73} := \{w - y + i = x + iy - 1 = 0\}$	$\ell_{74} := \{w - y + i = x - iy + 1 = 0\}$
$\ell_{75} := \{w - y + i = x + y + i = 0\}$	$\ell_{76} := \{w - y + i = x - y - i = 0\}$
$\ell_{77} := \{w - y - i = x + iy + 1 = 0\}$	$\ell_{78} := \{w - y - i = x - iy - 1 = 0\}$
$\ell_{79} := \{w - y - i = x + y - i = 0\}$	$\ell_{80} := \{w - y - i = x - y + i = 0\}$
$\ell_{81} := \{w - y + 1 = x + iy + i = 0\}$	$\ell_{82} := \{w - y + 1 = x - iy - i = 0\}$
$\ell_{83} := \{w - y + 1 = x + y + 1 = 0\}$	$\ell_{84} := \{w - y + 1 = x - y - 1 = 0\}$
$\ell_{85} := \{w - y - 1 = x + iy - i = 0\}$	$\ell_{86} := \{w - y - 1 = x - iy + i = 0\}$
$\ell_{87} := \{w - y - 1 = x + y - 1 = 0\}$	$\ell_{88} := \{w - y - 1 = x - y + 1 = 0\}$
$\ell_{89} := \{w - (1-i)y = x + (1+i) = 0\}$	$\ell_{90} := \{w - (1-i)y = x + (1-i) = 0\}$
$\ell_{91} := \{w - (1-i)y = x - (1-i) = 0\}$	$\ell_{92} := \{w - (1-i)y = x - (1+i) = 0\}$
$\ell_{93} := \{w - (1+i)y = x + (1+i) = 0\}$	$\ell_{94} := \{w - (1+i)y = x + (1-i) = 0\}$
$\ell_{95} := \{w - (1+i)y = x - (1-i) = 0\}$	$\ell_{96} := \{w - (1+i)y = x - (1+i) = 0\}$
$\ell_{97} := \{w + (1+i)x = y + (1+i) = 0\}$	$\ell_{98} := \{w + (1+i)x = y + (1-i) = 0\}$
$\ell_{99} := \{w + (1+i)x = y - (1-i) = 0\}$	$\ell_{100} := \{w + (1+i)x = y - (1+i) = 0\}$
$\ell_{101} := \{w + (1-i)x = y + (1+i) = 0\}$	$\ell_{102} := \{w + (1-i)x = y + (1-i) = 0\}$
$\ell_{103} := \{w + (1-i)x = y - (1-i) = 0\}$	$\ell_{104} := \{w + (1-i)x = y - (1+i) = 0\}$
$\ell_{105} := \{w - (1-i)x = y + (1+i) = 0\}$	$\ell_{106} := \{w - (1-i)x = y + (1-i) = 0\}$
$\ell_{107} := \{w - (1-i)x = y - (1-i) = 0\}$	$\ell_{108} := \{w - (1-i)x = y - (1+i) = 0\}$
$\ell_{109} := \{w - (1+i)x = y + (1+i) = 0\}$	$\ell_{110} := \{w - (1+i)x = y + (1-i) = 0\}$
$\ell_{111} := \{w - (1+i)x = y - (1-i) = 0\}$	$\ell_{112} := \{w - (1+i)x = y - (1+i) = 0\}$

TABLE 3.1. Lines on  $X$

-2	1	1	1	1	0	0	1	0	0	1	0	0	0	0	0	0	0	1	1	0	1
1	-2	1	1	0	1	0	0	1	0	0	0	1	1	0	0	0	1	0	0	1	0
1	1	-2	1	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0
1	1	1	-2	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0
1	0	0	0	-2	1	1	1	0	0	0	1	0	0	0	1	1	0	0	0	0	0
0	1	0	0	1	-2	1	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0
0	0	1	0	1	1	-2	0	0	1	0	0	1	1	0	0	0	0	0	0	1	0
1	0	0	0	1	0	0	-2	1	1	0	0	1	0	1	0	0	0	0	0	1	0
0	1	0	0	0	1	0	1	-2	1	0	1	0	0	0	0	1	0	0	0	0	0
0	0	1	0	0	0	1	1	1	-2	1	0	0	0	0	1	0	1	0	1	0	1
1	0	0	0	0	0	0	0	0	1	-2	1	1	1	0	0	1	0	0	0	0	0
0	0	0	1	1	0	0	0	1	0	1	-2	1	0	1	0	0	1	0	1	0	0
0	1	0	0	0	0	1	1	0	0	1	1	-2	0	0	1	0	0	1	0	0	0
0	1	0	0	0	0	1	0	0	0	1	0	0	-2	1	1	1	0	0	1	0	0
0	0	1	0	0	1	0	1	0	0	0	1	0	1	-2	1	0	1	0	0	0	1
0	0	0	1	1	0	0	0	0	1	0	0	1	1	1	-2	0	0	1	0	1	0
0	0	1	0	1	0	0	0	1	0	1	0	0	1	0	0	-2	1	1	0	1	0
0	1	0	0	0	0	0	0	0	1	0	1	0	0	1	0	1	-2	1	0	0	0
1	0	0	0	0	1	0	0	0	0	0	0	1	0	0	1	1	1	-2	0	0	1
1	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	-2	1	0
0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	1	1	0	0	1	-2	1
1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	1	-2

TABLE 3.2. Gram matrix  $N$  of  $S$ 

From the intersection numbers of the 112 lines, we can calculate their classes  $[\ell_i] \in S$ .

*Remark 3.1.* Any other choice of the 22 lines that span  $S$  will do for the calculation of  $\text{Aut}(X)$ . We have chosen and fixed (3.1) only in order to fix the notation of vectors in  $S^\vee$ .

*Remark 3.2.* Since these 112 lines are all defined over  $\mathbb{F}_9$ , every class  $v \in S$  is represented by a divisor defined over  $\mathbb{F}_9$ . More generally, Schütt [30] showed that a supersingular  $K3$  surface with Artin invariant 1 in characteristic  $p$  has a projective model defined over  $\mathbb{F}_p$ , and its Néron-Severi lattice is generated by the classes of divisors defined over  $\mathbb{F}_{p^2}$ .

**Proposition 3.3.** *We have  $h_0 = \frac{1}{28} \sum_{i=1}^{112} [\ell_i]$ .*

*Proof.* The number of  $\mathbb{F}_9$ -rational points on  $X$  is 280. For each  $\mathbb{F}_9$ -rational point  $P$  of  $X$ , the tangent plane  $T_{X,P} \subset \mathbb{P}^3$  to  $X$  at  $P$  cuts out a union of four lines from  $X$ . Since each line contains ten  $\mathbb{F}_9$ -rational points, we have  $280 h_0 = 10 \sum [\ell_i]$ .  $\square$

As before, we let  $O(S)$  act on  $S$  from *right*, so that

$$O(S) = \{ T \in \mathrm{GL}_{22}(\mathbb{Z}) \mid T N {}^t T = N \}.$$

We also let the projective automorphism group  $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$  act on  $X$  from *right*. For each  $\tau \in \mathrm{PGU}_4(\mathbb{F}_9)$ , we can calculate its action  $\tau_*$  on  $S$  by looking at the permutation of the 112 lines induced by  $\tau$ .

**Example 3.4.** Consider the projective automorphism

$$\tau : [w : x : y : z] \mapsto [w : x : y : z] \begin{bmatrix} i & 0 & i & -1+i \\ 1 & 1-i & -1 & 0 \\ 1 & i & i & -i \\ 1 & -1 & -i & -1 \end{bmatrix}$$

of  $X$ . Then the images  $\ell_i^\tau$  of the lines  $\ell_i$  in (3.1) are

$$\begin{aligned} \ell_1^\tau &= \ell_{60}, \ell_2^\tau = \ell_{31}, \ell_3^\tau = \ell_{105}, \ell_4^\tau = \ell_{95}, \ell_5^\tau = \ell_{92}, \ell_6^\tau = \ell_{30}, \ell_7^\tau = \ell_{76}, \\ \ell_9^\tau &= \ell_{110}, \ell_{10}^\tau = \ell_{29}, \ell_{11}^\tau = \ell_6, \ell_{17}^\tau = \ell_{20}, \ell_{18}^\tau = \ell_{96}, \ell_{19}^\tau = \ell_{102}, \\ \ell_{21}^\tau &= \ell_{13}, \ell_{22}^\tau = \ell_{87}, \ell_{23}^\tau = \ell_{91}, \ell_{25}^\tau = \ell_{108}, \ell_{26}^\tau = \ell_{10}, \\ \ell_{27}^\tau &= \ell_{57}, \ell_{33}^\tau = \ell_{52}, \ell_{35}^\tau = \ell_{51}, \ell_{49}^\tau = \ell_{59}. \end{aligned}$$

Therefore the action  $\tau_*$  on  $S$  is given by  $v \mapsto v T_\tau$ , where  $T_\tau$  is the matrix whose row vectors are given in Table 3.3

We put the representation

$$(3.2) \quad \tau \mapsto T_\tau$$

of  $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$  to  $O^+(S)$  in the computer memory. It turns out to be faithful. On the other hand,  $\mathrm{Aut}(X, h_0)$  is just the stabilizer subgroup in  $\mathrm{Aut}(X)$  of  $h_0 \in S$ . Therefore we confirm the following fact ([28, Section 8, Proposition 3]):

**Proposition 3.5.** *The action of  $\mathrm{Aut}(X)$  on  $S$  is faithful.*

From now on, we regard  $\mathrm{Aut}(X)$  as a subgroup of  $O^+(S)$ , and write  $v \mapsto v^\gamma$  instead of  $v \mapsto v^{\gamma*}$  for the action  $\gamma_*$  of  $\gamma \in \mathrm{Aut}(X)$  on  $S$ .

$[\ell_{60}]$	$=$	$[1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 1, 0, 0, 1]_S,$
$[\ell_{31}]$	$=$	$[1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0, -1, 0, 0, 0]_S,$
$[\ell_{105}]$	$=$	$[2, 2, 2, 3, -1, -1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, -1, 0]_S,$
$[\ell_{95}]$	$=$	$[-3, -2, -2, -3, 1, 1, 2, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, -1]_S,$
$[\ell_{92}]$	$=$	$[-1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, -1, -1, -1]_S,$
$[\ell_{30}]$	$=$	$[1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0]_S,$
$[\ell_{76}]$	$=$	$[0, -1, -1, -1, 0, -1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1]_S,$
$[\ell_{110}]$	$=$	$[-1, 0, 0, -1, 0, 0, 1, 0, 0, 0, 0, -1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0]_S,$
$[\ell_{29}]$	$=$	$[1, 1, 1, 1, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0]_S,$
$[\ell_6]$	$=$	$[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S,$
$[\ell_{20}]$	$=$	$[1, 1, 1, 1, 0, 0, 0, 0, 0, 0, -1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S,$
$[\ell_{96}]$	$=$	$[4, 2, 3, 4, -2, -3, -2, -1, -2, 0, 1, 0, -1, 0, -1, -1, -1, -1, -1, -1, 2, 0, 1]_S,$
$[\ell_{102}]$	$=$	$[-1, -1, -1, -1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, -1, 0, 0]_S,$
$[\ell_{13}]$	$=$	$[0, 1, 1, 1, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S,$
$[\ell_{87}]$	$=$	$[-3, -2, -3, -3, 2, 2, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, -1, 0, -1]_S,$
$[\ell_{91}]$	$=$	$[4, 2, 3, 3, -1, -2, -1, 0, -1, 0, 0, -1, -1, -1, -1, -1, -1, -1, -1, 1, 0, 1]_S,$
$[\ell_{108}]$	$=$	$[-2, -2, -2, -3, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0]_S,$
$[\ell_{10}]$	$=$	$[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_S,$
$[\ell_{57}]$	$=$	$[1, 2, 1, 2, -1, 0, -1, -1, 0, -1, 0, 1, 0, 0, 0, -1, 0, 1, 0, 0, -1, -1]_S,$
$[\ell_{52}]$	$=$	$[-1, 0, -1, -1, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -1, 0, -1]_S,$
$[\ell_{51}]$	$=$	$[1, 1, 1, 2, 0, 0, -1, -1, 0, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1]_S,$
$[\ell_{59}]$	$=$	$[2, 1, 2, 2, -1, -2, 0, 0, -1, 1, 0, -1, -1, 0, -1, 0, -1, -1, -1, 1, 1, 1]_S.$

TABLE 3.3. Row vectors of  $T_\tau$  in Example 3.44. EMBEDDING OF  $S$  INTO  $L$ 

Next we embed the Néron-Severi lattice  $S$  of  $X$  into the even unimodular hyperbolic lattice of rank 26, and calculate the walls of an  $\mathcal{R}_S$ -chamber.

Let  $T$  be the negative-definite root lattice of type  $2A_2$ . We fix a basis of  $T$  in such a way that the Gram matrix is equal to

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

When we use this basis, we write elements of  $T \otimes \mathbb{R}$  as  $[y_1, y_2, y_3, y_4]_T$ , while when we use its dual basis, we write as  $[\eta_1, \eta_2, \eta_3, \eta_4]_T^\vee$ . Elements of  $(S \oplus T) \otimes \mathbb{R}$  are written as

$$[x_1, \dots, x_{22} \mid y_1, \dots, y_4]$$

using the bases of  $S$  and  $T$ , or as

$$[\xi_1, \dots, \xi_{22} \mid \eta_1, \dots, \eta_4]^\vee$$

using the dual bases of  $S^\vee$  and  $T^\vee$ .

Consider the following vectors of  $S^\vee \oplus T^\vee$ :

$$\begin{aligned} a_1 &:= \frac{1}{3} [2, 2, 0, 0, 0, 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 0, 0, 1, 1, 0, 0, 0 \mid 1, 2, 0, 0], \\ a_2 &:= \frac{1}{3} [2, 0, 2, 0, 2, 1, 1, 0, 2, 1, 2, 1, 0, 2, 2, 1, 1, 0, 1, 0, 0, 0 \mid 0, 0, 1, 2]. \end{aligned}$$

We define  $\alpha_1, \alpha_2 \in (S \oplus T)^\vee / (S \oplus T)$  by

$$\alpha_1 := a_1 \bmod (S \oplus T), \quad \alpha_2 := a_2 \bmod (S \oplus T).$$

Then  $\alpha_1$  and  $\alpha_2$  are linearly independent in  $(S \oplus T)^\vee / (S \oplus T) \cong \mathbb{F}_3^4$ . Since

$$q_{S \oplus T}(\alpha_1) = q_{S \oplus T}(\alpha_2) = q_{S \oplus T}(\alpha_1 + \alpha_2) = 0,$$

the vectors  $\alpha_1$  and  $\alpha_2$  generate a maximal isotropic subgroup of  $q_{S \oplus T}$ . Therefore, by [21, Proposition 1.4.1], the submodule

$$L := (S \oplus T) + \langle a_1 \rangle + \langle a_2 \rangle$$

of  $S^\vee \oplus T^\vee$  is an even unimodular overlattice of  $S \oplus T$  into which  $S$  and  $T$  are primitively embedded.

By construction,  $L$  is hyperbolic of rank 26. We choose  $\mathcal{P}_L$  to be the connected component that contains  $\mathcal{P}_S$ . Then, by means of the roots of  $L$ , we obtain a decomposition of  $\mathcal{P}_S$  into  $\mathcal{R}_S$ -chambers.

The order of  $O(T)$  is 288, while the order of  $O(q_T)$  is 8. It is easy to check that the natural homomorphism  $O(T) \rightarrow O(q_T)$  is surjective. Therefore we obtain the following from Proposition 2.7:

**Proposition 4.1.** *The action of  $O^+(S)$  on  $S \otimes \mathbb{R}$  preserves  $\mathcal{R}_S$ .*

We put

$$\begin{aligned} w_0 &:= [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \mid -1, -1, -1, -1] \\ &= [1, 1 \mid 1, 1, 1, 1]^\vee. \end{aligned}$$

Note that the projection  $w_{0S} \in S^\vee$  of  $w_0$  to  $S^\vee$  is equal to  $h_0$ .

Since  $(w_0, w_0)_L = 0$  and  $(w_0, h_0)_L > 0$ , we see that  $w_0$  is on the boundary of the closure of  $\mathcal{P}_L$  in  $L \otimes \mathbb{R}$ .

**Proposition 4.2.** *The vector  $w_0$  is a Weyl vector, and the  $\mathcal{R}_L$ -chamber  $D_L(w_0)$  is  $S$ -nondegenerate. The  $\mathcal{R}_S$ -chamber*

$$D_{S0} := D_L(w_0) \cap (S \otimes \mathbb{R})$$

*contains  $w_{0S} = h_0$  in its interior.*



[illegible]

We show that  $w_0$  satisfies the conditions (i) and (ii) given after Definition 2.5. By Proposition 2.2, in order to verify the condition (i), it is enough to show that the function  $Q : U^\perp \rightarrow \mathbb{Z}$  given by

$$Q(\lambda) := (h_0, -\frac{2 + (\lambda, \lambda)_L}{2}w_0 + w'_0 + \lambda)_L$$

$$LR(w_0, S) = \{ r \in LR(w_0) \mid (r_S, r_S)_S < 0 \}$$

*Remark 4.3.* There exist exactly four vectors  $\lambda \in U^\perp$  such that  $Q(\lambda) = 0$ . They correspond to the Leech roots  $r \in LR(w_0)$  such that  $r = r_T$ .

From the surjectivity of  $O(T) \rightarrow O(q_T)$  and Proposition 2.7, we obtain the following:

**Corollary 4.4.** *The action of  $\text{Aut}(X, h_0)$  on  $S \otimes \mathbb{R}$  preserves  $D_{S_0}$  and  $\widetilde{\mathcal{W}}(D_{S_0})$ .*

**Proposition 4.5.** *The maps  $r \mapsto r_S$  and  $r_S \mapsto (r_S)^\perp_S$  induce bijections*

$$LR(w_0, S) \cong \widetilde{\mathcal{W}}(D_{S_0}) \cong \mathcal{W}(D_{S_0}).$$

The action of  $\text{Aut}(X, h_0)$  decomposes  $\widetilde{\mathcal{W}}(D_{S_0})$  into the three orbits

$$\widetilde{W}_{112} := \widetilde{W}(D_{S0})_{[1,-2]}, \quad \widetilde{W}_{648} := \widetilde{W}(D_{S0})_{[2,-4/3]} \quad \text{and} \quad \widetilde{W}_{5184} := \widetilde{W}(D_{S0})_{[3,-2/3]}$$

of cardinalities 112, 648 and 5184, respectively, where

$$\widetilde{\mathcal{W}}(D_{S_0})_{[a,n]} := \{ r_S \in \widetilde{\mathcal{W}}(D_{S_0}) \mid (r_S, h_0)_S = a, (r_S, r_S)_S = n \}.$$

The set  $\widetilde{W}_{112}$  coincides with the set of the classes  $[\ell_i]$  of lines contained in  $X$ :

$$\widetilde{W}_{112} = \{[\ell_1], [\ell_2], \dots, [\ell_{112}]\}.$$

The sets  $\widetilde{W}_{648}$  and  $\widetilde{W}_{5184}$  are the orbits of

$$b_1 = \frac{1}{3} [-1, 0, -1, 0, 2, 1, 1, 0, 2, 1, -1, 1, 0, -1, -1, 1, 1, 0, 1, 0, 0, 0]_S \in \widetilde{W}_{648}$$

and

$$b_2 := \frac{1}{3} [0, 1, -1, 0, 2, 0, 2, 1, 1, 0, 0, -1, 2, 1, 0, 1, 1, -1, 0, 0, 0, 0]_S \in \widetilde{W}_{5184}$$

by the action of  $\text{Aut}(X, h_0)$ , respectively.

*Proof.* We have calculated the finite set  $LR(w_0, S)$  in the proof of Proposition 4.2. We have also stored the classes  $[\ell_i]$  of the 112 lines and the action of  $\text{Aut}(X, h_0)$  on  $S$  in the computer memory. Thus the assertions of Proposition 4.5 are verified by a direct computation, except for the fact that, for any  $r \in LR(w_0, S)$ , the hyperplane  $(r_S)^\perp_S$  actually bounds  $D_{S0}$ . This is proved by showing that the point

$$p := h_0 - \frac{(h_0, r_S)_S}{(r_S, r_S)_S} r_S$$

on  $(r_S)^\perp_S$  satisfies  $(p, r')_L > 0$  for any  $r' \in LR(w_0, S) \setminus \{r\}$ .  $\square$

Since Proposition 3.3 implies that the interior point  $h_0$  of  $D_{S0}$  is determined by  $\widetilde{W}_{112}$  and since  $O(T) \rightarrow O(q_T)$  is surjective, we obtain the following from Proposition 2.7:

**Corollary 4.6.** *For  $\gamma \in O^+(S)$ , the following are equivalent: (i) the interior of  $D_{S0}^\gamma$  has a common point with  $D_{S0}$ , (ii)  $D_{S0}^\gamma = D_{S0}$ , (iii)  $\widetilde{W}_{112}^\gamma = \widetilde{W}_{112}$ , (iv)  $h_0^\gamma = h_0$ , and (v)  $h_0^\gamma \in D_{S0}$ .*

In particular, we obtain the following:

**Corollary 4.7.** *If  $\gamma \in \text{Aut}(X)$  satisfies  $h_0^\gamma \in D_{S0}$ , then  $\gamma$  is in  $\text{Aut}(X, h_0)$ .*

## 5. THE AUTOMORPHISMS $g_1$ AND $g_2$

In order to find automorphisms  $\gamma \in \text{Aut}(X)$  such that  $h_0^\gamma \notin D_{S0}$ , we search for polarizations of degree 2 that are located on the walls  $(b_1)^\perp_S$  and  $(b_2)^\perp_S$ .

We fix terminologies and notation. For a vector  $v \in S$ , we denote by  $\mathcal{L}_v \rightarrow X$  a line bundle defined over  $\mathbb{F}_9$  whose class is  $v$  (see Remark 3.2). We say that a vector  $h \in S$  is a *polarization* of degree  $d$  if  $(h, h)_S = d$  and the complete linear system  $|\mathcal{L}_h|$  is nonempty and has no fixed components. If  $h$  is a polarization, then  $|\mathcal{L}_h|$  has no base-points by [29, Corollary 3.2] and hence defines a morphism

$$\Phi_h : X \rightarrow \mathbb{P}^N,$$

where  $N = \dim |\mathcal{L}_h|$ .

A polynomial in  $\mathbb{F}_9[w, x, y]$  is said to be of *normal form* if its degree with respect to  $w$  is  $\leq 3$ . For each polynomial  $G \in \mathbb{F}_9[w, x, y]$ , there exists a unique polynomial  $\overline{G}$  of normal form such that

$$G \equiv \overline{G} \pmod{(w^4 + x^4 + y^4 + 1)}.$$

We say that  $\overline{G}$  is the *normal form* of  $G$ . For any  $d \in \mathbb{Z}$ , the vector space  $H^0(X, \mathcal{L}_{dh_0})$  over  $\mathbb{F}_9$  is naturally identified with the vector subspace

$$\Gamma(d) := \{ G \in \mathbb{F}_9[w, x, y] \mid G \text{ is of normal form with total degree } \leq d \}$$

of  $\mathbb{F}_9[w, x, y]$ . For an ideal  $J$  of  $\mathbb{F}_9[w, x, y]$ , we put

$$\Gamma(d, J) := \Gamma(d) \cap J.$$

A basis of  $\Gamma(d, J)$  is easily obtained by a Gröbner basis of  $J$ . Let  $\ell_i$  be a line contained in  $X$ . We denote by  $I_i \subset \mathbb{F}_9[w, x, y]$  the affine defining ideal of  $\ell_i$  in  $\mathbb{P}^3$  (see Table 3.1), and put

$$I_i^{(\nu)} := I_i^\nu + (w^4 + x^4 + y^4 + 1) \subset \mathbb{F}_9[w, x, y]$$

for nonnegative integers  $\nu$ . Suppose that  $v \in S$  is written as

$$(5.1) \quad v = d h_0 - \sum_{i=1}^{112} a_i [\ell_i],$$

where  $a_i$  are nonnegative integers. Then there exists a natural isomorphism

$$H^0(X, \mathcal{L}_v) \cong \Gamma(d, \bigcap_{i=1}^{112} I_i^{(a_i)})$$

with the property that, for another vector  $v' = d' h_0 - \sum_{i=1}^{112} a'_i [\ell_i]$  with  $a'_i \in \mathbb{Z}_{\geq 0}$ , the multiplication homomorphism

$$H^0(X, \mathcal{L}_v) \times H^0(X, \mathcal{L}_{v'}) \rightarrow H^0(X, \mathcal{L}_{v+v'})$$

is identified with

$$\Gamma(d, \bigcap_{i=1}^{112} I_i^{(a_i)}) \times \Gamma(d', \bigcap_{i=1}^{112} I_i^{(a'_i)}) \rightarrow \Gamma(d + d', \bigcap_{i=1}^{112} I_i^{(a_i + a'_i)})$$

given by  $(\overline{G}, \overline{G'}) \mapsto \overline{GG'}$ .

Proposition 1.1 in Introduction is an immediate consequence of the following:

**Proposition 5.1.** *Consider the vectors*

$$m_1 := [-1, 0, -1, -1, 2, 2, 1, 1, 2, 0, -1, 1, 1, -1, 0, 1, 0, 0, 1, -1, 0, 0]_S \quad \text{and}$$

$$m_2 := [2, 2, 1, 2, 1, -1, 1, 1, 1, 1, 0, -1, 0, 0, 0, 0, 0, -1, -1, 1, 0, 1]_S$$

of  $S$ . Then each  $m_\nu$  is a polarization of degree 2. If we choose a basis of the vector space  $H^0(X, \mathcal{L}_{m_\nu})$  appropriately, the morphism  $\Phi_{m_\nu} : X \rightarrow \mathbb{P}^2$  associated with  $|\mathcal{L}_{m_\nu}|$  coincides with the morphism  $\phi_\nu : X \rightarrow \mathbb{P}^2$  given in the statement of Proposition 1.1.

*Proof.* We have  $(m_\nu, m_\nu)_S = 2$ . By the method described in [39, Section 4.1], we see that  $m_\nu$  is a polarization; namely, we verify that the sets

$$\{ v \in S \mid (v, v)_S = -2, (v, m_\nu)_S < 0, (v, h_0)_S > 0 \}$$

and

$$\{ v \in S \mid (v, v)_S = 0, (v, m_\nu)_S = 1 \}$$

are both empty. Since

$$(5.2) \quad m_1 = 3h_0 - ([\ell_{21}] + [\ell_{22}] + [\ell_{50}] + [\ell_{63}] + [\ell_{65}] + [\ell_{88}])$$

and

$$(5.3) \quad m_2 = 5h_0 - ([\ell_1] + [\ell_3] + [\ell_6] + [\ell_{18}] + [\ell_{35}] + [\ell_{74}] + [\ell_{90}] + [\ell_{92}] + [\ell_{110}] + [\ell_{111}]),$$

the vector spaces  $H^0(X, \mathcal{L}_{m_1})$  and  $H^0(X, \mathcal{L}_{m_2})$  are identified with the subspaces

$$\begin{aligned} \Gamma_1 &:= \Gamma(3, I_{21} \cap I_{22} \cap I_{50} \cap I_{63} \cap I_{65} \cap I_{88}) \quad \text{and} \\ \Gamma_2 &:= \Gamma(5, I_1 \cap I_3 \cap I_6 \cap I_{18} \cap I_{35} \cap I_{74} \cap I_{90} \cap I_{92} \cap I_{110} \cap I_{111}) \end{aligned}$$

of  $\mathbb{F}_9[w, x, y]$ , respectively. We calculate a basis of  $\Gamma_\nu$  by means of Gröbner bases of the ideals  $I_i$ . The set  $\{F_{\nu 0}, F_{\nu 1}, F_{\nu 2}\}$  of polynomials in Table 1.1 is just a basis of  $\Gamma_\nu$  thus calculated.  $\square$

*Remark 5.2.* The polarizations  $m_1$  and  $m_2$  in Proposition 5.1 are located on the hyperplanes  $(b_1)_S^\perp$  and  $(b_2)_S^\perp$  bounding  $D_{S0}$ , respectively, where  $b_1 \in \widetilde{W}_{648}$  and  $b_2 \in \widetilde{W}_{5184}$  are given in Proposition 4.5.

We now prove Proposition 1.2.

*Proof.* The set  $\text{Exc}(\phi_\nu)$  of the classes of  $(-2)$ -curves contracted by  $\phi_\nu : X \rightarrow \mathbb{P}^2$  is calculated by the method described in [39, Section 4.2]. We first calculate the set

$$R_\nu^+ := \{ v \in S \mid (v, v)_S = -2, (v, m_\nu)_S = 0, (v, h_0)_S > 0 \}.$$

It turns out that every element of  $R_\nu^+$  is written as a linear combination with coefficients in  $\mathbb{Z}_{\geq 0}$  of elements  $l \in R_\nu^+$  such that  $(l, h_0)_S = 1$ . Hence we have

$$\text{Exc}(\phi_\nu) = \{ l \in R_\nu^+ \mid (l, h_0)_S = 1 \}.$$

The *ADE*-type of the root system  $\text{Exc}(\phi_\nu)$  is equal to  $6A_1 + 4A_2$  for  $\nu = 1$  and  $A_1 + A_2 + 2A_3 + 2A_4$  for  $\nu = 2$ . Thus the assertion on the *ADE*-type of the singularities of  $Y_\nu$  is proved. Moreover we have proved that all  $(-2)$ -curves contracted by  $\phi_\nu : X \rightarrow \mathbb{P}^2$  are lines. See Tables 5.1 and 5.2, in which the lines  $\ell_{k_1}, \dots, \ell_{k_r}$  contracted by  $\phi_\nu$  to a singular point  $P$  of type  $A_r$  are indicated in such an order that  $(\ell_{k_j}, \ell_{k_{j+1}})_S = 1$  holds for  $j = 1, \dots, r-1$ .

The defining equation  $f_\nu = 0$  of the branch curve  $B_\nu \subset \mathbb{P}^2$  is calculated by the method given in [39, Section 5]. We calculate a basis of the vector space  $H^0(X, \mathcal{L}_{3m_\nu})$  of dimension 11 using (5.2), (5.3) and Gröbner bases of  $I_i^{(3)}$ . Note that

$\ell_{37}$	$\mapsto$	$[1 : 1 - i : 1 - i]$	$(A_1\text{-point})$
$\ell_{23}$	$\mapsto$	$[1 : 1 + i : -(1 + i)]$	$(A_1\text{-point})$
$\ell_{62}$	$\mapsto$	$[1 : -(1 + i) : 0]$	$(A_1\text{-point})$
$\ell_{102}$	$\mapsto$	$[1 : -(1 - i) : 0]$	$(A_1\text{-point})$
$\ell_{68}$	$\mapsto$	$[1 : 1 + i : 1 + i]$	$(A_1\text{-point})$
$\ell_{112}$	$\mapsto$	$[1 : 1 - i : -(1 - i)]$	$(A_1\text{-point})$
$\ell_{49}, \ell_{29}$	$\mapsto$	$[1 : 1 : -i]$	$(A_2\text{-point})$
$\ell_{73}, \ell_{60}$	$\mapsto$	$[1 : 1 : i]$	$(A_2\text{-point})$
$\ell_{18}, \ell_{10}$	$\mapsto$	$[0 : 1 : -1]$	$(A_2\text{-point})$
$\ell_{16}, \ell_{99}$	$\mapsto$	$[0 : 1 : 1]$	$(A_2\text{-point})$

TABLE 5.1. Lines contracted by  $\phi_1 : X \rightarrow \mathbb{P}^2$ 

$\ell_{43}$	$\mapsto$	$[0 : 1 : 0]$	$(A_1\text{-point})$
$\ell_{76}, \ell_{94}$	$\mapsto$	$[1 : -1 : 0]$	$(A_2\text{-point})$
$\ell_{22}, \ell_{49}, \ell_{20}$	$\mapsto$	$[1 : -1 : 1]$	$(A_3\text{-point})$
$\ell_7, \ell_5, \ell_{103}$	$\mapsto$	$[1 : -1 : -1]$	$(A_3\text{-point})$
$\ell_{10}, \ell_2, \ell_4, \ell_{91}$	$\mapsto$	$[1 : 0 : 1]$	$(A_4\text{-point})$
$\ell_{33}, \ell_{36}, \ell_{72}, \ell_{83}$	$\mapsto$	$[1 : 0 : -1]$	$(A_4\text{-point})$

TABLE 5.2. Lines contracted by  $\phi_2 : X \rightarrow \mathbb{P}^2$ 

the ten normal forms  $M_{\nu,1}, \dots, M_{\nu,10}$  of the cubic monomials of  $F_{\nu 0}, F_{\nu 1}, F_{\nu 2}$  are contained in  $H^0(X, \mathcal{L}_{3m_\nu})$ . We choose a polynomial  $G_\nu \in H^0(X, \mathcal{L}_{3m_\nu})$  that is not contained in the linear span of  $M_{\nu,1}, \dots, M_{\nu,10}$ . In the vector space  $H^0(X, \mathcal{L}_{6m_\nu})$  of dimension 38, the 39 normal forms of the monomials of  $G_\nu, F_{\nu 0}, F_{\nu 1}, F_{\nu 2}$  of weighted degree 6 with weight  $\deg G_\nu = 3$  and  $\deg F_{\nu j} = 1$  have a non-trivial linear relation. Note that this linear relation is quadratic with respect to  $G_\nu$ . Completing the square and re-choosing  $G_\nu$  appropriately, we confirm that

$$\overline{G_\nu^2 + f_\nu(F_{\nu 0}, F_{\nu 1}, F_{\nu 2})} = 0$$

holds. Hence  $Y_\nu$  is defined by  $y^2 + f_\nu(x_0, x_1, x_2) = 0$ .  $\square$

*Remark 5.3.* In order to obtain a defining equation of  $B_\nu$  with coefficients in  $\mathbb{F}_3$ , we have to choose the basis  $F_{\nu 0}, F_{\nu 1}, F_{\nu 2}$  of  $\Gamma_\nu = H^0(X, \mathcal{L}_{m_\nu})$  carefully. See [39, Section 6.10] for the method.

*Remark 5.4.* The polynomial

$$G_1 = G_{1(0)}(x, y) + G_{1(1)}(x, y)w + G_{1(2)}(x, y)w^2 + G_{1(3)}(x, y)w^3$$

is given in Table 5.3. The polynomial  $G_2$  is too large to be presented in the paper (see [40]).

$$\begin{aligned}
G_{1(0)} &= -(1-i) + (1+i)x + (1+i)y + ix^2 - (1+i)xy - (1+i)y^2 - xy^2 \\
&\quad + (1+i)y^3 - (1-i)x^4 - (1+i)x^3y - xy^3 - (1-i)y^4 - (1-i)x^5 - x^3y^2 \\
&\quad - ix^2y^3 - (1-i)xy^4 + (1+i)y^5 - (1-i)x^6 + x^5y + ix^4y^2 - (1-i)x^3y^3 \\
&\quad + (1+i)x^2y^4 - ixy^5 + (1-i)y^6 + (1+i)x^7 + x^4y^3 + (1+i)x^3y^4 \\
&\quad + ixy^6 - iy^7 + ix^8 + (1+i)x^7y + ix^6y^2 - ix^5y^3 + (1-i)x^4y^4 + x^2y^6 \\
&\quad + ixy^7 - iy^8 - (1+i)x^9 - ix^8y - (1-i)x^7y^2 - (1+i)x^6y^3 + ix^5y^4 \\
&\quad + (1-i)x^4y^5 - (1+i)x^3y^6 - (1-i)x^2y^7 - (1-i)xy^8 - (1+i)y^9 \\
G_{1(1)} &= (1-i) + (1-i)x - (1+i)x^2 - xy + iy^2 + x^3 - (1-i)x^2y - xy^2 \\
&\quad + (1+i)x^4 + (1-i)xy^3 - (1-i)y^4 - x^5 + x^4y + xy^4 - (1+i)y^5 \\
&\quad + (1+i)x^5y - (1+i)xy^5 + y^6 - ix^7 - x^6y + x^5y^2 - (1-i)x^4y^3 \\
&\quad + (1-i)x^2y^5 + (1-i)xy^6 - (1-i)y^7 - ix^8 + (1+i)x^7y - ix^6y^2 \\
&\quad - ix^5y^3 - (1-i)x^4y^4 + (1+i)x^3y^5 - (1-i)x^2y^6 + (1-i)xy^7 + (1-i)y^8 \\
G_{1(2)} &= (1-i) - (1+i)x - (1+i)xy + y^2 - (1-i)x^3 - (1+i)x^2y - ixy^2 \\
&\quad - (1+i)y^3 + x^4 - (1+i)x^3y + xy^3 - y^4 - x^5 - x^4y - xy^4 - y^5 \\
&\quad - ix^6 + x^4y^2 + ix^3y^3 + (1+i)xy^5 - y^6 - (1-i)x^7 - ix^6y - ix^5y^2 \\
&\quad - ix^4y^3 - (1+i)x^3y^4 - (1+i)x^2y^5 + (1+i)xy^6 - (1+i)y^7 \\
G_{1(3)} &= (1+i)x - (1-i)y - (1-i)x^2 - (1-i)xy + (1+i)y^2 - x^3 - x^2y \\
&\quad - xy^2 + y^3 - (1-i)x^4 - ix^3y + (1-i)xy^3 - iy^4 + ix^5 + x^4y \\
&\quad + (1+i)x^3y^2 + (1+i)x^2y^3 + (1+i)xy^4 + (1+i)y^5 - x^6 - (1-i)x^5y \\
&\quad + (1+i)x^4y^2 + ix^3y^3 - (1-i)x^2y^4 - (1+i)xy^5 + (1+i)y^6
\end{aligned}$$

TABLE 5.3. Polynomial  $G_1$ 

**Proposition 5.5.** *Let  $g_1$  and  $g_2$  be the involutions of  $X$  defined in Theorem 1.3. Then the action  $g_{\nu*}$  on  $S$  is given by  $v \mapsto vA_\nu$ , where  $A_\nu$  is the matrix given in Tables 5.4 and 5.5.*

*Proof.* Recall that  $\text{Exc}(\phi_\nu)$  is the set of the classes of  $(-2)$ -curves contracted by  $\phi_\nu : X \rightarrow \mathbb{P}^2$ . Suppose that  $\gamma_1, \dots, \gamma_r \in \text{Exc}(\phi_\nu)$  are the classes of  $(-2)$ -curves that are contracted to a singular point  $P \in \text{Sing}(B_\nu)$  of type  $A_r$ . We index them in such a way that  $(\gamma_\nu, \gamma_{\nu+1})_S = 1$  holds for  $\nu = 1, \dots, r-1$ . Then  $g_{\nu*}$  interchanges  $\gamma_\nu$  and  $\gamma_{r+1-\nu}$ . Let  $V(P) \subset S \otimes \mathbb{Q}$  denote the linear span of the invariant vectors  $\gamma_\nu + \gamma_{r+1-\nu}$ . Then the eigenspace of  $g_{\nu*}$  on  $S \otimes \mathbb{Q}$  with eigenvalue 1 is equal to

$$\langle m_\nu \rangle \oplus \bigoplus_{P \in \text{Sing}(B_\nu)} V(P),$$

and the eigenspace with eigenvalue  $-1$  is its orthogonal complement.  $\square$

Using the matrix representations  $A_\nu$  of  $g_{\nu*}$ , we verify the following facts:

- (1) The eigenspace of  $g_{\nu*}$  with eigenvalue 1 is contained in  $(b_\nu)_S^\perp$ . In particular, we have  $b_\nu^{g_\nu} = -b_\nu$ .

-1	0	0	0	1	1	1	0	1	0	-1	0	0	0	0	1	1	0	1	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
1	1	0	1	1	0	0	0	1	0	0	1	0	-1	-1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	-1	-1	-1	0	0	0	0	0	0
0	1	0	1	1	1	0	0	1	0	-1	0	0	-1	0	1	0	0	1	-1	0	0
0	-1	0	-1	0	-1	1	0	-1	1	0	-1	0	1	0	1	0	-1	0	1	1	1
-2	-1	-2	-2	2	1	1	1	2	1	0	1	1	0	0	1	1	0	0	0	0	-1
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	1	0	0	1	0	0	0	0	0	-1	1	0	-1	0	1	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	1	1	0	0	1	0	-1	1	0	-1	0	0	0	1	1	-1	-1	0
-1	0	-1	0	1	1	0	0	2	0	0	2	1	-1	0	0	1	1	1	-1	-1	-1
1	1	1	1	0	0	1	0	0	1	-1	-1	-1	0	-1	0	0	-1	0	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	-1	-1	-1	0	0	0	0	0	0	0	0	0
2	2	2	3	0	-1	-1	-1	0	0	0	1	0	-1	-1	0	0	0	0	0	-1	0
-2	-1	-2	-2	1	1	0	1	0	1	1	1	1	0	0	1	0	0	0	0	-1	-1
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	3	0	0	-1	-1	1	0	0	1	-1	-1	-1	-1	0	0	0	0	-1	0
1	1	1	1	0	0	0	0	0	0	-1	0	0	-1	0	0	-1	0	0	0	0	0

TABLE 5.4. The matrix  $A_1$ 

- (2) The vector  $h_0^{g_\nu}$  is equal to the image of  $h_0$  by the reflection into the wall  $(b_\nu)^\perp_S$ , that is  $h_0^{g_1} = h_0 + 3b_1$  and  $h_0^{g_2} = h_0 + 9b_2$  hold.

Since  $\text{Aut}(X, h_0)$  acts on each of  $\widetilde{W}_{648}$  and  $\widetilde{W}_{5184}$  transitively, we obtain the following:

**Corollary 5.6.** *For any  $r_S \in \widetilde{W}_{648} \cup \widetilde{W}_{5184}$ , there exists  $\tau \in \text{Aut}(X, h_0)$  such that*

$$h_0^{g_\nu^\tau} = h_0 + c_\nu r_S$$

*holds, where  $\nu = 1$  and  $c_1 = 3$  if  $r_S \in \widetilde{W}_{648}$  while  $\nu = 2$  and  $c_2 = 9$  if  $r_S \in \widetilde{W}_{5184}$ .*

1	1	-1	0	2	0	2	1	1	0	0	-1	2	1	0	1	1	-1	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	2	-1	0	4	0	4	2	2	0	0	-2	4	2	0	2	2	-2	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	2	-2	0	4	0	4	2	2	0	0	-1	4	2	0	2	2	-2	0	0	0	0
0	0	0	0	-1	-1	0	0	0	1	1	0	0	1	0	0	0	0	-1	1	0	0
0	2	0	1	1	0	2	0	1	0	0	-1	2	1	0	1	1	-1	0	0	0	0
4	2	3	3	-1	-2	-10	-1	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	1	0	1
-3	-1	-4	-3	4	2	3	2	2	0	0	0	3	1	1	2	2	0	1	-1	0	-1
2	2	1	2	1	-2	2	1	0	1	0	-2	1	1	-1	1	0	-2	-1	1	1	1
0	2	-2	0	4	1	4	2	2	0	0	-2	4	2	0	2	2	-2	0	0	0	0
-1	0	-1	-1	1	1	0	0	1	-1	0	1	1	0	0	0	1	1	1	-1	-1	-1
1	2	0	1	2	0	2	0	0	-1	0	-1	2	1	0	1	1	-1	0	0	0	0
1	1	1	1	0	0	0	0	0	0	-1	-1	-1	0	0	0	0	0	0	0	0	0
0	0	-1	-1	2	0	3	2	1	1	0	-2	2	1	0	1	1	-2	0	0	1	1
-2	-1	-2	-2	1	2	1	1	1	0	-1	0	1	0	1	1	0	0	1	-1	0	0
1	2	0	1	1	-1	1	0	1	0	1	0	2	1	0	0	1	0	0	0	-1	0
0	2	0	1	2	0	2	1	1	0	0	-1	2	1	0	1	1	-1	-1	0	0	-1
0	1	1	1	0	0	0	0	1	0	0	1	0	0	0	-1	0	0	-1	0	-1	-1
4	5	1	4	3	-1	2	1	1	-1	0	-2	3	1	-1	1	1	-2	0	0	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

TABLE 5.5. The matrix  $A_2$ 

## 6. PROOF OF THEOREM 1.3

We denote by

$$G := \langle \text{Aut}(X, h_0), g_1, g_2 \rangle$$

the subgroup of  $\text{Aut}(X)$  generated by  $\text{Aut}(X, h_0)$ ,  $g_1$  and  $g_2$ . Note that the action of  $\text{Aut}(X)$  on  $S$  preserves the set of nef classes.

**Theorem 6.1.** *If  $v \in S$  is nef, there exists  $\gamma \in G$  such that  $v^\gamma \in D_{S_0}$ .*

*Proof.* Let  $\gamma \in G$  be an element such that  $(v^\gamma, h_0)_S$  attains

$$\min\{(v^{\gamma'}, h_0)_S \mid \gamma' \in G\}.$$



We show that  $(v^\gamma, r_S)_S \geq 0$  holds for any  $r_S \in \widetilde{\mathcal{W}}(D_{S_0})$ . If  $r_S \in \widetilde{W}_{112}$ , then  $r_S = [\ell_i]$  for some line  $\ell_i \subset X$ , and hence  $(v^\gamma, r_S)_S \geq 0$  holds because  $v^\gamma$  is nef. Suppose that  $r_S \in \widetilde{W}_{648} \cup \widetilde{W}_{5184}$ . By Corollary 5.6, there exists  $\tau \in \text{Aut}(X, h_0)$  such that  $h_0^{g_\nu^\tau} = h_0 + c_\nu r_S$  holds, where  $\nu = 1$  and  $c_1 = 3$  if  $r_S \in \widetilde{W}_{648}$  while  $\nu = 2$  and  $c_2 = 9$  if  $r_S \in \widetilde{W}_{5184}$ . Since  $\gamma\tau^{-1}g_\nu \in G$ , we have

$$(v^\gamma, h_0)_S \leq (v^{\gamma\tau^{-1}g_\nu}, h_0)_S = (v^\gamma, h_0^{g_\nu^\tau})_S = (v^\gamma, h_0)_S + c_\nu(v^\gamma, r_S)_S.$$

Therefore  $(v^\gamma, r_S)_S \geq 0$  holds.  $\square$

The properties (1), (2), (3) of  $D_{S_0}$  stated in Introduction follow from Corollaries 4.4, 4.6, 4.7 and Theorem 6.1. We now prove Theorem 1.3.

*Proof.* By Corollary 4.7, it is enough to show that, for any  $\gamma \in \text{Aut}(X)$ , there exists  $\gamma' \in G$  such that  $h_0^{\gamma\gamma'} \in D_{S_0}$  holds. Since  $h_0^\gamma$  is nef, this follows from Theorem 6.1.  $\square$

As a geometric consequence, we present the following:

**Corollary 6.2.** *The group  $\text{Aut}(X)$  acts on the set of  $(-2)$ -curves on  $X$  transitively with the stabilizer group of order  $13063680/112 = 116640$ .*

*Remark 6.3.* See Sterk [46] for a general result on the action of the automorphism group on the set of  $(-2)$ -curves.

## 7. THE FERMAT QUARTIC POLARIZATIONS FOR $g_1$ AND $g_2$

A polarization  $h \in S$  of degree 4 is said to be a *Fermat quartic polarization* if, by choosing an appropriate basis of  $H^0(X, \mathcal{L}_h)$ , the morphism  $\Phi_h : X \rightarrow \mathbb{P}^3$  associated with  $|\mathcal{L}_h|$  induces an automorphism of  $X \subset \mathbb{P}^3$ . It is obvious that  $h_0^\gamma$  is a Fermat quartic polarization for any  $\gamma \in \text{Aut}(X)$ . Conversely, if  $h$  is a Fermat quartic polarization, then the pull-back of  $h_0$  by the automorphism  $\Phi_h$  of  $X$  is  $h$ . Therefore the set of Fermat quartic polarizations is the orbit of  $h_0$  by the action of  $\text{Aut}(X)$  on  $S$ . Consider the Fermat quartic polarizations

$$\begin{aligned} h_1 &:= h_0^{g_1} = h_0 A_1 = [0, 1, 0, 1, 2, 1, 1, 0, 2, 1, -1, 1, 0, -1, -1, 1, 1, 0, 1, 0, 0, 0]_S, \\ h_2 &:= h_0^{g_2} = h_0 A_2 = [1, 4, -2, 1, 6, 0, 6, 3, 3, 0, 0, -3, 6, 3, 0, 3, 3, -3, 0, 0, 0, 0]_S. \end{aligned}$$

Using the equalities

$$\begin{aligned} h_1 &= 6h_0 - ([\ell_3] + [\ell_6] + [\ell_8] + [\ell_{14}] + [\ell_{15}] + [\ell_{17}] + [\ell_{19}] + \\ (7.1) \quad &+ [\ell_{22}] + [\ell_{31}] + [\ell_{34}] + [\ell_{63}] + [\ell_{70}] + [\ell_{79}] + [\ell_{92}]), \end{aligned}$$

$$\begin{aligned} h_2 &= 15h_0 - (3[\ell_3] + 4[\ell_6] + [\ell_{13}] + [\ell_{14}] + 3[\ell_{18}] + \\ (7.2) \quad &+ [\ell_{22}] + [\ell_{26}] + [\ell_{27}] + 2[\ell_{35}] + [\ell_{44}] + 2[\ell_{50}] + \\ &+ 3[\ell_{92}] + [\ell_{93}] + [\ell_{106}] + [\ell_{108}] + 3[\ell_{111}]), \end{aligned}$$

we obtain another description of the involutions  $g_1$  and  $g_2$ .

**Theorem 7.1.** *Let  $(w, x, y)$  be the affine coordinates of  $\mathbb{P}^3$  with  $z = 1$ , and let*

$$H_{1j}(w, x, y) = H_{1j0}(x, y) + H_{1j1}(x, y)w + H_{1j2}(x, y)w^2 + H_{1j3}(x, y)w^3$$

*be polynomials given in Table 7.1. Then the rational map*

$$(7.3) \quad (w, x, y) \mapsto [H_{10} : H_{11} : H_{12} : H_{13}] \in \mathbb{P}^3$$

*gives the involution  $g_1$  of  $X$ .*

*Proof.* We put

$$Z := \{3, 6, 8, 14, 15, 17, 19, 22, 31, 34, 63, 70, 79, 92\},$$

which is the set of indices of lines on  $X$  that appear in the right-hand side of (7.1).

The polynomials  $H_{10}, H_{11}, H_{12}, H_{13}$  form a basis of the vector space

$$H^0(X, \mathcal{L}_{h_1}) \cong \Gamma(6, \bigcap_{i \in Z} I_i).$$

(See Section 5 for the notation.) We can easily verify that

$$H_{10}^4 + H_{11}^4 + H_{12}^4 + H_{13}^4 \equiv 0 \pmod{(w^4 + x^4 + y^4 + 1)}$$

holds. Hence the rational map (7.3) induces an automorphism  $g'$  of  $X$ . We prove  $g' = g_1$  by showing that the action  $g'_*$  of  $g'$  on  $S$  is equal to the action  $v \mapsto vA_1$  of  $g_1$ . We homogenize the polynomials  $H_{1j}$  to  $\tilde{H}_{1j}(w, x, y, z)$  so that  $g'$  is given by

$$[w : x : y : z] \mapsto [\tilde{H}_{10} : \tilde{H}_{11} : \tilde{H}_{12} : \tilde{H}_{13}].$$

Let  $\ell_k$  be a line on  $X$  whose index  $k$  is not in  $Z$ . We calculate a parametric representation

$$[u : v] \mapsto [l_{k0} : l_{k1} : l_{k2} : l_{k3}]$$

of  $\ell_k$  in  $\mathbb{P}^3$ , where  $u, v$  are homogeneous coordinates of  $\mathbb{P}^1$  and  $l_{k\nu}$  are homogeneous linear polynomials of  $u, v$ . We put

$$L_{1j}^{(k)} := \tilde{H}_{1j}(l_{k0}, l_{k1}, l_{k2}, l_{k3})$$

for  $j = 0, \dots, 3$ , which are homogeneous polynomials of  $u, v$ . Let  $M^{(k)}$  be the greatest common divisor of  $L_{10}^{(k)}, L_{11}^{(k)}, L_{12}^{(k)}, L_{13}^{(k)}$  in  $\mathbb{F}_9[u, v]$ . Then

$$\rho_k : [u : v] \mapsto [L_{10}^{(k)}/M^{(k)} : L_{11}^{(k)}/M^{(k)} : L_{12}^{(k)}/M^{(k)} : L_{13}^{(k)}/M^{(k)}]$$

is a parametric representation of the image of  $\ell_k$  by  $g'$ . (If  $k \in Z$ , then  $L_{1j}^{(k)}$  are constantly equal to 0.) Pulling back the defining homogeneous ideal of  $\ell_{k'}$  by  $\rho_k$ , we can calculate the intersection number  $([\ell_k]^{g'}, [\ell_{k'}])_S$ . Since the classes  $[\ell_k]$  with  $k \notin Z$  span  $S \otimes \mathbb{Q}$ , we can calculate the action  $g'_*$  of  $g'$  on  $S$ , which turns out to be equal to  $v \mapsto vA_1$ .  $\square$

*Remark 7.2.* We have a similar list of polynomials  $H_{20}, H_{21}, H_{22}, H_{23}$  that gives the involution  $g_2$ . They are, however, too large to be presented in the paper (see [40]).

$$\begin{aligned}
H_{100} &= -1 - (1-i)x - (1-i)x^2 - (1+i)y^2 - ix^3 - (1-i)xy^2 + x^4 - (1-i)x^3y \\
&\quad + (1+i)x^2y^2 + (1+i)xy^3 - (1-i)y^4 - (1-i)x^5 + (1+i)x^4y - (1-i)x^3y^2 - ix^2y^3 \\
&\quad - ixy^4 + iy^5 + ix^6 - x^5y - (1+i)x^4y^2 - (1+i)x^3y^3 + (1+i)xy^5 + (1-i)y^6 \\
H_{101} &= (1+i) + x + (1-i)y - ix^2 - (1+i)xy + iy^2 + ix^3 - x^2y \\
&\quad - ixy^2 + y^3 + x^3y + (1-i)x^2y^2 - (1-i)xy^3 + (1-i)y^4 \\
&\quad + x^5 + ix^4y + x^3y^2 - ix^2y^3 + (1-i)xy^4 - (1-i)y^5 \\
H_{102} &= i + x + (1+i)y + x^2 + (1+i)y^2 + (1+i)x^3 - x^2y \\
&\quad - (1+i)xy^2 + iy^3 + (1+i)x^4 + (1-i)x^2y^2 + xy^3 + (1+i)y^4 \\
H_{103} &= (1-i) - (1-i)x + (1-i)y - (1+i)x^2 - (1-i)xy + (1+i)x^3 - (1+i)y^3 \\
\hline
H_{110} &= -i + ix + y - (1+i)x^2 + xy - (1-i)y^2 - x^3 - (1-i)x^2y + (1+i)xy^2 - y^3 \\
&\quad + (1+i)x^4 - ix^3y - (1-i)x^2y^2 + xy^3 + (1-i)y^4 - (1+i)x^5 + (1-i)x^4y + ix^2y^3 \\
&\quad - (1+i)xy^4 - (1+i)y^5 - ix^6 + (1+i)x^4y^2 + (1+i)x^3y^3 + (1-i)xy^5 - (1-i)y^6 \\
H_{111} &= -(1-i) + x + (1+i)y - (1+i)x^2 - ixy - iy^2 + (1-i)x^3 \\
&\quad - ix^2y - y^3 + (1+i)x^4 + (1-i)x^3y - x^2y^2 + (1+i)xy^3 \\
&\quad - iy^4 + (1-i)x^5 + ix^4y + (1-i)x^2y^3 + (1-i)xy^4 - y^5 \\
H_{112} &= -1 + (1+i)y + x^2 - (1-i)xy - (1+i)y^2 - x^2y + (1+i)xy^2 \\
&\quad - (1+i)y^3 + (1-i)x^4 + (1+i)x^3y - (1+i)x^2y^2 - xy^3 - (1+i)y^4 \\
H_{113} &= (1+i) - x + y + x^2 - iy^2 - (1-i)x^3 + ix^2y - (1-i)xy^2 - iy^3 \\
\hline
H_{120} &= (1+i) + (1+i)x + (1+i)y + (1-i)x^2 + y^2 + (1+i)x^3 + (1+i)x^2y \\
&\quad - ixy^2 - y^3 - (1-i)x^3y + (1-i)x^2y^2 - (1+i)xy^3 + (1-i)y^4 \\
&\quad + (1-i)x^5 - ix^4y + (1-i)x^3y^2 - (1+i)x^2y^3 + ixy^4 - y^5 + x^6 \\
&\quad - (1+i)x^5y - (1-i)x^4y^2 + x^3y^3 - ix^2y^4 - (1-i)xy^5 + (1-i)y^6 \\
H_{121} &= i + x + xy - (1+i)y^2 + x^3 - (1+i)x^2y - (1-i)xy^2 + (1+i)y^3 + x^4 - (1-i)x^3y \\
&\quad - (1-i)x^2y^2 + (1+i)xy^3 - (1-i)y^4 - (1-i)x^5 + (1+i)x^3y^2 + (1+i)x^2y^3 + (1-i)y^5 \\
H_{122} &= (1-i) - x - (1+i)y + ix^2 - (1-i)xy - (1+i)y^2 - x^3 - (1-i)xy^2 \\
&\quad - iy^3 - (1+i)x^4 - (1-i)x^3y - (1+i)x^2y^2 - xy^3 + (1+i)y^4 \\
H_{123} &= 1 - (1+i)x + (1-i)y + x^2 + ixy + iy^2 - (1+i)x^3 + (1-i)x^2y - (1+i)xy^2 + (1+i)y^3 \\
\hline
H_{130} &= -(1-i) + ix + (1+i)y - (1+i)x^2 + (1-i)xy + (1-i)y^2 + x^3 - (1+i)x^2y + ixy^2 + iy^3 \\
&\quad - (1+i)x^4 + ix^3y + x^2y^2 - (1+i)y^4 + (1+i)x^5 - (1-i)x^4y + (1-i)x^3y^2 - x^2y^3 \\
&\quad - (1+i)y^5 - (1+i)x^6 - (1-i)x^5y - (1+i)x^4y^2 + ix^3y^3 + ix^2y^4 + ixy^5 + (1+i)y^6 \\
H_{131} &= -1 - x + (1+i)y - (1-i)x^2 + (1+i)xy - iy^2 - (1+i)x^3 - ix^2y - xy^2 + iy^3 \\
&\quad - x^4 - x^3y + xy^3 - (1+i)y^4 - (1+i)x^5 + x^4y + (1-i)x^3y^2 - ix^2y^3 + (1+i)xy^4 \\
H_{132} &= (1+i) + ix + y - x^2 + xy + y^2 + ix^3 - (1-i)x^2y \\
&\quad - (1+i)xy^2 - (1-i)x^4 - x^2y^2 - ixy^3 - (1-i)y^4 \\
H_{133} &= i - y + x^2 + (1+i)xy - (1-i)y^2
\end{aligned}$$

TABLE 7.1. Polynomials  $H_{1j}$

*Remark 7.3.* The polynomials  $H_{10}, H_{11}, H_{12}, H_{13}$  are found by the following method. Let  $H'_0, H'_1, H'_2, H'_3$  be an arbitrary basis of  $\Gamma(6, \cap_{i \in \mathbb{Z}} I_i) \cong H^0(X, \mathcal{L}_{h_1})$ . Then the normal forms of the quartic monomials of  $H'_0, H'_1, H'_2, H'_3$  are subject to a linear relation of the following form (see [33, n. 3] or [39, Theorem 6.11]):

$$\sum_{i,j=0}^3 a_{ij} \overline{H'_i H'^3_j} = 0,$$

where the coefficients  $a_{ij} \in \mathbb{F}_9$  satisfy  $a_{ji} = a_{ij}^3$  and  $\det(a_{ij}) \neq 0$ ; that is, the matrix  $(a_{ij})$  is non-singular Hermitian. We search for  $B \in \mathrm{GL}_3(\mathbb{F}_9)$  such that

$$(a_{ij}) = B {}^t B^{(3)}$$

holds, where  $B^{(3)}$  is obtained from  $B$  by applying  $x \mapsto x^3$  to the entries, and put

$$(H''_0, H''_1, H''_2, H''_3) = (H'_0, H'_1, H'_2, H'_3)B.$$

Then  $H''_0, H''_1, H''_2, H''_3$  satisfy

$$H''_0{}^4 + H''_1{}^4 + H''_2{}^4 + H''_3{}^4 \equiv 0 \pmod{(w^4 + x^4 + y^4 + 1)}.$$

Therefore  $(w, x, y) \mapsto [H''_0 : H''_1 : H''_2 : H''_3]$  induces an automorphism  $g''$  of  $X$ . Using the method described in the proof of Theorem 7.1, we calculate the matrix  $A''$  such that the action  $g''_*$  of  $g''$  on  $S$  is given by  $v \mapsto vA''$ . Next we search for  $\tau \in \mathrm{PGU}_4(\mathbb{F}_9)$  such that  $A''T_\tau$  is equal to  $A_1$ , where  $T_\tau \in \mathrm{O}^+(S)$  is the matrix representation of  $\tau$ . Then the polynomials

$$(H_{10}, H_{11}, H_{12}, H_{13}) := (H''_0, H''_1, H''_2, H''_3)\tau$$

have the required property.

*Remark 7.4.* We have calculated the images of the  $\mathbb{F}_9$ -rational points of  $X$  by the morphisms  $\psi_\nu : X \rightarrow Y_\nu$  and  $g_\nu : X \rightarrow X$ , and confirmed that they are compatible (see [40]).

## 8. GENERATORS OF $\mathrm{O}^+(S)$

Let  $F \in \mathrm{O}^+(S)$  denote the isometry of  $S$  obtained from the Frobenius action  $\phi$  of  $\mathbb{F}_9$  over  $\mathbb{F}_3$  on  $X$ . Calculating the action of  $\phi$  on the lines  $(\ell_1^\phi = \ell_6, \ell_2^\phi = \ell_5, \ell_3^\phi = \ell_8, \ell_4^\phi = \ell_7, \dots)$ , we see that  $F$  is given  $v \mapsto vA_F$ , where  $A_F$  is the matrix presented in Table 8.1. Since  $h_0^F = h_0$ , we have  $D_{S_0}^F = D_{S_0}$  by Corollary 4.6.

**Proposition 8.1.** *The automorphism group  $\mathrm{Aut}(D_{S_0}) \subset \mathrm{O}^+(S)$  of  $D_{S_0}$  is the split extension of  $\langle F \rangle \cong \mathbb{Z}/2\mathbb{Z}$  by  $\mathrm{Aut}(X, h_0)$ .*

*Proof.* Since we have calculated the representation (3.2) of  $\mathrm{Aut}(X, h_0)$  into  $\mathrm{O}^+(S)$ , we can verify that  $F \notin \mathrm{Aut}(X, h_0)$ . Therefore it is enough to show that the order of  $\mathrm{Aut}(D_{S_0})$  is equal to 2 times  $|\mathrm{Aut}(X, h_0)|$ . Since  $|\mathrm{PGU}_4(\mathbb{F}_9)|$  is equal to 4 times  $|\mathrm{PSU}_4(\mathbb{F}_9)|$ , this follows from [15, Lemma 2.1] (see also [9, p. 52]).  $\square$

0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	0	-1	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	-1	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	-1	-1	-2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
2	1	1	2	-1	-2	-1	-1	-1	0	1	0	0	0	-1	0	0	0	0	1	0
0	0	0	0	1	1	0	1	1	0	-1	0	0	-1	0	0	0	0	-1	0	0
2	2	2	2	-1	0	-1	-1	0	-1	0	0	-1	0	0	-1	0	0	0	0	-1
0	1	1	0	0	1	1	0	0	0	-1	-1	0	0	0	0	0	0	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
2	1	1	2	0	0	-1	0	0	-1	0	1	0	-1	0	-1	-1	0	0	0	-1
0	0	0	0	0	0	1	0	-1	0	0	-1	0	1	0	1	0	-1	0	0	1
-3	-2	-2	-3	1	1	1	1	1	1	0	0	1	0	1	1	1	1	1	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
4	3	3	4	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0

TABLE 8.1. Frobenius action on  $S$ 

Since  $([\ell_1], [\ell_1])_S = -2$ , the reflection  $s_1 : S \otimes \mathbb{R} \rightarrow S \otimes \mathbb{R}$  into the hyperplane  $([\ell_1])_S^\perp$  is contained in  $O^+(S)$ . In the same way as the proof of Theorem 1.3, we obtain the following:

**Theorem 8.2.** *The autochronous orthogonal group  $O^+(S)$  of the Néron-Severi lattice  $S$  of  $X$  is generated by  $\text{Aut}(X, h_0) = \text{PGU}_4(\mathbb{F}_9)$ ,  $g_1$ ,  $g_2$ ,  $F$  and  $s_1$ .*

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